

# Baxter operators for arbitrary spin

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## Abstract

We construct Baxter operators for the homogeneous closed XXX spin chain with the quantum space carrying infinite or finite dimensional  $sl_2$  representations. All algebraic relations of Baxter operators and transfer matrices are deduced uniformly from Yang-Baxter relations of the local building blocks of these operators. This results in a systematic and very transparent approach where the cases of finite and infinite dimensional representations are treated in analogy. Simple relations between the Baxter operators of both cases are obtained. We represent the quantum spaces by polynomials and build the operators from elementary differentiation and multiplication operators. We present compact explicit formulae for the action of Baxter operators on polynomials.

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## 1 Introduction

The quantum inverse scattering method (QISM) [1–4] is the modern approach to the theory of integrable systems. In the framework of QISM, eigenstates  $|v_1, \dots, v_k\rangle$  of the set of commuting operators are obtained by the algebraic Bethe ansatz (ABA) method as excitations over a formal vacuum state and the spectral problem is reduced to the set of algebraic Bethe equations for the parameters  $v_j$ .

The basic tools for analyzing a quantum spin chain are the monodromy matrix  $\mathbb{T}(u)$  constructed as the product of the L-operators referring to the sites of the chain

$$\mathbb{T}(u) \equiv L_1(u)L_2(u)\dots L_n(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \quad (1.1)$$

and the transfer matrix  $t(u)$  involving the family of commuting operators,  $t(u)t(v) = t(v)t(u)$  constructed as the matrix trace

$$t(u) \equiv \text{tr } \mathbb{T}(u) = A(u) + D(u). \quad (1.2)$$

Usually there exists some reference state  $|0\rangle$  playing the role of lowest weight vector:

$$B(u)|0\rangle = 0 ; A(u)|0\rangle = \Delta_+(u)|0\rangle ; D(u)|0\rangle = \Delta_-(u)|0\rangle ,$$

where  $\Delta_{\pm}(u)$  are functions of spectral parameter  $u$ .

In the ABA approach one shows that the vector  $|v_1, \dots, v_k\rangle = C(v_1) \cdots C(v_k)|0\rangle$  is an eigenvector of the operator  $t(u)$  with the eigenvalue  $\tau_k(u)$ :

$$t(u)|v_1, \dots, v_k\rangle = \tau_k(u)|v_1, \dots, v_k\rangle \quad ; \quad \tau_k(u) = \Delta_+(u) \frac{Q^{(k)}(u+1)}{Q^{(k)}(u)} + \Delta_-(u) \frac{Q^{(k)}(u-1)}{Q^{(k)}(u)} \quad (1.3)$$

if the parameters  $v_i$  obey the Bethe equations:

$$Q^{(k)}(v_i+1) \Delta_+(v_i) + Q^{(k)}(v_i-1) \Delta_-(v_i) = 0 . \quad (1.4)$$

All information about parameters  $v_i$  is accumulated in a polynomial  $Q^{(k)}(u) = (u-v_1) \cdots (u-v_k)$ .

One can obtain the Bethe equation from the formula for  $\tau(u)$  by taking residue at  $u = v_i$  and using the fact that the polynomial  $\tau(u)$  is regular at this point. Finally we see that equations (1.3) and (1.4) are equivalent to the Baxter equation for the polynomial  $Q^{(k)}(u)$ ,

$$\tau_k(u) Q^{(k)}(u) = \Delta_+(u) Q^{(k)}(u+1) + \Delta_-(u) Q^{(k)}(u-1) . \quad (1.5)$$

There exists an alternative approach to the solution of the model – the method of Q-operators.

In this approach the whole problem is reduced to the construction of the Baxter Q-operator. In general it is an operator  $Q(u)$  with the properties [5]:

- commutativity

$$[Q(u), Q(v)] = 0 ; [Q(u), t(v)] = 0$$

- finite-difference Baxter equation

$$t(u) \cdot Q(u) = \Delta_+(u) Q(u+1) + \Delta_-(u) Q(u-1) . \quad (1.6)$$

Note that in this approach the meaning of the function  $Q^{(k)}(u) = (u-v_1) \cdots (u-v_k)$  is the polynomial eigenvalue of the Q-operator

$$Q(u) |v_1, \dots, v_k\rangle = Q^{(k)}(u) |v_1, \dots, v_k\rangle . \quad (1.7)$$

The concept of Q-operators has been introduced by Baxter analyzing the eight-vertex model [5].

Such operators have been studied in a number of particular models [6–20], where we quote here only some papers where models with the simplest symmetry group of the rank one are considered.

Using a Q-operator one can perform the transformation to the separated variable representation [21–23], where the eigenfunctions appear in factorized form.

In spite of such a variety of results and long history there is so far no commonly accepted general scheme of Q-operator construction. Also the relation between different approaches to the problem has not been discussed exhaustively.

In this paper we pursue two purposes.

At first we carry out a systematization of the results of [24–26] where the case of infinite dimensional representations of the symmetry group with rank one has been considered.

We simplify that construction significantly proving all defining properties of Q-operators uniformly. Namely we derive each global relation (involving the chain operators, Q-operators, transfer matrices) from a corresponding local relation for their building blocks (R-operators and L-operators).

In other words we pull down the whole construction systematically to the local level such that properties of the global objects become more transparent. Secondly using this construction we consider finite dimensional representation in the quantum space and obtain corresponding Q-operators. That is we propose a solution of the problem for any integer or half-integer spin as well. It is important that the Q-operators for representations of both types are connected intrinsically in our approach. We restrict ourselves to the XXX homogeneous closed spin chain in this paper in order to make our argumentation clearer.

In [27] Baxter operators have been considered for the closed spin- $\frac{1}{2}$  chain. We decided to devote a separate paper [28] to the comparison of our approach to the ones developed there.

The plan of the paper is the following. The first part is devoted to a generic situation: the spin parameter  $\ell$  is an arbitrary complex number and representations are infinite-dimensional.

In a first step we construct all needed local objects – the L-operators and the general R-operators. The important tool on this stage is the construction of the R-operator in a factorized form. In our formulation operators are written in terms of canonical pairs  $z, \partial : [\partial, z] = 1$  and representation spaces are spanned by monomials in  $z$ . This should not lead to misunderstandings compared to some other papers where the notations  $\hat{a}, \hat{a}^\dagger$  are preferred instead.

Next we distract for a while from the systematic exposition and demonstrate the factorization of R-operators at work. We show that all needed ingredients for the construction of a Q-operator are present on this stage already: we construct a Q-operator and derive a useful formula for its action on the generating function of monomials. This construction of Q-operator does not explain its origin and has some disadvantages from the technical point of view because the proof of commutativity is not so simple.

So we return to the general line and proceed to the construction of the global objects of the chain. The general transfer matrix  $T_s(u)$  is obtained by replacing in the transfer matrix expression at each site  $k$  the L-matrix by the general Yang-Baxter R-operator acting on the tensor product of representation modules with spins  $\ell, s$  and taking trace in the representation space  $s$ . Like the ordinary transfer matrix all these operators are generating functions of the set of commuting operators of the quantum spin chain. It turns out that the global objects inherit factorization properties from the local objects: the general transfer matrix  $T_s(u)$  is factorized into a product of simpler operators, the Q-operators. At this moment the Q-operators are assigned their natural place in a general picture.

From this point of view the general transfer matrix  $T_s(u)$  is a one-parametric set of Baxter Q-operators. It is convenient to construct Baxter operators of simpler structure which can be understood as particular cases or restrictions of the former.

In the second part of the paper using this construction we consider finite dimensional representation in the quantum space and obtain corresponding Q-operators. That is we propose the construction for any integer or half-integer spin as well. We formulate explicitly the intrinsic connection between the Q-operators for representations of both types.

According to our ideology we start from the general R-operator and restrict it to finite-dimensional invariant subspace. The restricted R-operators are our local building blocks for construction of appropriate general transfer matrices and Baxter operators in the cases of integer or half-integer spin representation in the quantum space. The derivation of factorization of the general transfer matrix into the product of Q-operators follows step by step the corresponding calculation in the generic spin case. The same is true for the other properties of Baxter operators. Then we establish the connections between compact spin Q-operators and limits of Baxter operators for generic spin. We show that a careful calculation of limits with Q-operators for infinite-dimensional representations results in the appropriate Q-operators for finite-dimensional representations. This leads us to fairly simple, compact formulae for the Baxter operators for integer or half-integer spin.

## 2 Local objects: quantum L-operator and general R-operator

We consider first the operators representing the local building units of the chains. Actually the L-operator contains the local information about the system and the  $\mathbb{R}$ -operators are derived therefrom. On the other hand the  $\mathbb{R}$ -operators are more convenient as building units acting on the tensor product of quantum and auxiliary spaces carrying arbitrary representations. The particular case of spin  $\frac{1}{2}$  representation in one of these spaces leads us back to the L-matrix. The  $\mathbb{R}$ -operators provide us the starting point from which the different versions of Baxter operators can be obtained.

### 2.1 L-operators

We consider spin chains with  $sl_2$  symmetry algebra and use the functional representation of the algebra in the space of polynomials of one complex variable  $\mathbb{C}[z]$ . Fixing the generic complex number  $\ell$  and representing generators of the algebra as differential operators of the first order

$$S = z\partial - \ell, \quad S_- = -\partial, \quad S_+ = z^2\partial - 2\ell z \quad (2.1)$$

we provide  $\mathbb{C}[z]$  with the structure of a Verma module with lowest weight  $-\ell$  which we denote in the following by  $\mathbb{U}_{-\ell}$ . We keep this unusual sign for remembering that we are working with lowest weight vectors instead of highest weight vectors.

The module  $\mathbb{U}_{-\ell}$  is the infinite-dimensional functional space with the basis  $\{1, z, z^2, z^3, \dots\}$  and there is no invariant subspace in the module for generic  $\ell$ , in other words it is irreducible. An invariant finite-dimensional subspace  $\mathbb{V}_n$  appears for special values  $\ell = \frac{n}{2}, n = 0, 1, 2, 3, \dots$ . It is the  $(n+1)$ -dimensional irreducible submodule with the basis  $\{1, z, \dots, z^n\}$ . The infinite-dimensional quotient module  $\mathbb{U}_{-\frac{n}{2}+1} = \mathbb{U}_{-\frac{n}{2}}/\mathbb{V}_n$  with basis  $\{z^{n+1}, z^{n+2}, \dots\}$  is also irreducible. Its lowest weight is  $\frac{n}{2} + 1$ .

Representing generators as differential operators is very convenient since it allows to describe finite-dimensional (*compact spin*) and infinite-dimensional (*non-compact spin*) representations at once. Following this idea further will allow us to construct Q-operators for both types of representations in the quantum space in a compact and clear way.

We recover the  $sl_2$ -generators  $\mathbf{s}, \mathbf{s}_{\pm}$  in fundamental representation from expressions for generators in generic representations (2.1) for  $\ell = \frac{1}{2}$  with the basis  $\mathbf{e}_1 = S_+ \cdot 1 = -z$ ,  $\mathbf{e}_2 = 1$  in the known Pauli-matrix form

$$\mathbf{s} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{s}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{s}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where we used the standard definition for the matrix of an operator:  $A\mathbf{e}_i = \sum_k \mathbf{e}_k A_{ki}$ .

We define the L-operator as

$$L(u) \equiv u \cdot \mathbb{1} \otimes \mathbb{1} + 2 \cdot S \otimes \mathbf{s} + S_- \otimes \mathbf{s}_+ + S_+ \otimes \mathbf{s}_-$$

in terms of generators of the algebra (2.1). It acts in  $\mathbb{U}_{-\ell} \otimes \mathbb{C}^2$  and depends on two parameters: spin  $\ell$  and spectral parameter  $u$ . It respects Yang-Baxter equation in the space  $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{U}_{-\ell}$  for Yang R-matrix

$$R_{ij,nm}(u-v) \cdot L_{ns}(u) \cdot L_{mp}(v) = L_{is}(v) \cdot L_{jp}(u) \cdot R_{sp,nm}(u-v) \quad (2.2)$$

where  $i, j, \dots = 1, 2$  and  $R_{ij,nm}(u) = u \cdot \delta_{in} \delta_{jm} + \delta_{im} \delta_{jn}$ . This equation also referred to as the fundamental commutation relation contains in a compact form all relations of underlying Yangian

symmetry algebra. Taking into account the expressions for the generators the L-operator can be written as a matrix  $2 \times 2$  with operational elements acting in the space  $\mathbb{U}_{-\ell}$

$$L(u) = \begin{pmatrix} u - \ell + z\partial & -\partial \\ z^2\partial - 2\ell z & u + \ell - z\partial \end{pmatrix}. \quad (2.3)$$

There exists the useful factorized representation

$$L(u_1, u_2) = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} u_1 & -\partial \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}. \quad (2.4)$$

We have introduced the parameters  $u_1$  and  $u_2$ :  $u_1 \equiv u - \ell - 1$ ,  $u_2 \equiv u + \ell$  instead of  $u$  and  $\ell$  because they are very convenient for our purposes.

## 2.2 General R-operator

In order to avoid misunderstandings we distinguish two versions of the general Yang-Baxter operators acting in the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  by the notations  $R_{12}$  and  $\mathbb{R}_{12}$ . The former does not contain the permutation operator  $P_{12}$ :  $P_{12} \psi(z_1, z_2) = \psi(z_2, z_1)$  whereas the latter does, so they are related by

$$\mathbb{R}_{12} = P_{12} R_{12}.$$

In this section we use only R-operator notations however in the subsequent sections we will see that  $\mathbb{R}$ -operator notations are very natural for certain purposes.

The general R-operator acts on the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  and is defined as the  $sl_2$ -symmetric solution of the following relation (*RLL-relation*)

$$R_{12}(u_1, u_2|v_1, v_2) L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, v_2) L_2(u_1, u_2) R_{12}(u_1, u_2|v_1, v_2) \quad (2.5)$$

where

$$u_1 = u - \ell_1 - 1, \quad u_2 = u + \ell_1; \quad v_1 = v - \ell_2 - 1, \quad v_2 = v + \ell_2. \quad (2.6)$$

The operator  $R$  depends on the difference of spectral parameters  $u - v$  and two spins  $\ell_1, \ell_2$ , therefore the defining relation can be rewritten as

$$R(u - v|\ell_1, \ell_2) L_1(u) L_2(v) = L_1(v) L_2(u) R(u - v|\ell_1, \ell_2). \quad (2.7)$$

Roughly speaking the operator  $R$  interchanges parameters  $u_1, u_2$  with  $v_1, v_2$  in the product of L-operators.

$$L_1(u_1, u_2) L_2(v_1, v_2) \xrightarrow{R(u_1, u_2|v_1, v_2)} L_1(v_1, v_2) L_2(u_1, u_2)$$

It is useful to focus on this parameter exchange further and to consider more R-operators related to other exchange operations on the set  $(u_1, u_2, v_1, v_2)$ . In this set of operators there are those which interchange adjacent parameters. These are the most elementary ones because they are the building blocks for all other operators of parameter permutations. Thus the complex problem of solving the general Yang-Baxter equation reduces to a set of simpler ones. This idea has been carried out in the paper [26]. For our current purposes we do not need the most elementary operators of permutations and we work with some composite operators constructed from these elementary ones. The main reason for this is that the elementary operators are not well defined in the space  $\mathbb{C}[z]$  and one would have to consider a larger space, whereas the composite operators which we are going to work with here are well defined in the space  $\mathbb{C}[z]$ . In order to make the previous statements explicit we collect here some useful formulae.

Let us define the power of the derivative operator for any  $\alpha$  by

$$\partial^\alpha \equiv \frac{1}{z^\alpha} \frac{\Gamma(z\partial + 1)}{\Gamma(z\partial + 1 - \alpha)}. \quad (2.8)$$

Certainly, if  $\alpha$  takes values  $1, 2, \dots$  we obtain the familiar multiple derivative.

All expected properties of this operator can be easily proven using this definition.

- *commutativity and group property:*  $\partial^\alpha \partial^\beta = \partial^\beta \partial^\alpha = \partial^{\alpha+\beta}$  ;  $\partial^0 = \mathbf{1}$
- *differentiation rule*  $\partial^\alpha z = z \partial^\alpha + \alpha \partial^{\alpha-1}$
- *star-triangle relation :*  $\partial^\alpha z^{\alpha+\beta} \partial^\beta = z^\beta \partial^{\alpha+\beta} z^\alpha$
- *connection with  $\Gamma$ -functions:*  $z^\beta \partial^{\alpha+\beta} z^\alpha = \frac{\Gamma(z\partial+1+\alpha)}{\Gamma(z\partial+1-\beta)}$

We define the operators  $R_{12}^1$  and  $R_{12}^2$  acting in the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  by the following relations

$$R_{12}^1 L_1(u_1, u_2) L_2(v_1, v_2) = L_1(v_1, u_2) L_2(u_1, v_2) R_{12}^1 \quad (2.9)$$

$$R_{12}^2 L_1(u_1, u_2) L_2(v_1, v_2) = L_1(u_1, v_2) L_2(v_1, u_2) R_{12}^2 \quad (2.10)$$

To avoid misunderstanding we notice that upper indices 1,2 distinguish our two operators and lower indices usually show in which spaces the operators act nontrivially. In the generic situation  $\ell_1, \ell_2 \in \mathbb{C}$  the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  is isomorphic to the space  $\mathbb{C}[z_1, z_2]$  of polynomials of two variables  $z_1$  and  $z_2$  and the solutions of these equations have the form

$$\begin{aligned} R_{12}^1(u_1|v_1, v_2) &= z_{21}^{v_2-v_1} \partial_2^{u_1-v_1} z_{21}^{u_1-v_2} = \frac{\Gamma(z_{21}\partial_2 + u_1 - v_2 + 1)}{\Gamma(z_{21}\partial_2 + v_1 - v_2 + 1)} \\ R_{12}^2(u_1, u_2|v_2) &= z_{12}^{u_2-u_1} \partial_1^{u_2-v_2} z_{12}^{u_1-v_2} = \frac{\Gamma(z_{12}\partial_1 + u_1 - v_2 + 1)}{\Gamma(z_{12}\partial_1 + u_1 - u_2 + 1)}. \end{aligned} \quad (2.11)$$

It is evident that the operator  $R^k$  commutes with  $z_k$ :  $R^1 z_1 = z_1 R^1$  ;  $R^2 z_2 = z_2 R^2$ .

There are two ways to interchange parameters  $(u_1, u_2, v_1, v_2) \rightarrow (v_1, v_2, u_1, u_2)$  in the product of two L-operators by interchanging the ordering in the pairs  $u_1, v_1$  and  $u_2, v_2$ . Correspondingly the R-operator can be factorized in two ways as follows

$$R(u_1, u_2|v_1, v_2) = R^1(u_1|v_1, u_2) R^2(u_1, u_2|v_2) = R^2(v_1, u_2|v_2) R^1(u_1|v_1, v_2) \quad (2.12)$$

This can be checked with the star-triangle relation.

Now we would like to illustrate the factorization of the general R-operator by simple and transparent pictures. The operator  $R^1$  interchanges  $u_1$  and  $v_1$ :

$$L_1(u_1, u_2) L_2(v_1, v_2) \xrightarrow{R^1(u_1|v_1, v_2)} L_1(v_1, u_2) L_2(u_1, v_2)$$

and the operator  $R^2$  interchanges  $u_2$  and  $v_2$ :

$$L_1(u_1, u_2) L_2(v_1, v_2) \xrightarrow{R^2(u_1, u_2|v_2)} L_1(u_1, v_2) L_2(v_1, u_2)$$

The operator  $R^1 R^2$  interchanges parameters  $u_1, v_1$  and  $u_2, v_2$  in two steps so that the factorization indicated above is the condition of commutativity for the diagram

$$\begin{array}{ccc}
L_1(u_1, u_2) L_2(v_1, v_2) & \xrightarrow{R^2(u_1, u_2|v_2)} & L_1(u_1, v_2) L_2(v_1, u_2) \\
& \searrow R(u_1, u_2|v_1, v_2) & \downarrow R^1(u_1|v_1, u_2) \\
& & L_1(v_1, v_2) L_2(u_1, u_2)
\end{array}$$

Two equivalent ways to interchange parameters can be depicted by commutative diagram

$$\begin{array}{ccccc}
L_1(u_1, u_2) L_2(v_1, v_2) & \xrightarrow{R^2(u_1, u_2|v_2)} & L_1(u_1, v_2) L_2(v_1, u_2) & & \\
& \searrow R^1(u_1|v_1, v_2) & & \searrow R^1(u_1|v_1, u_2) & \\
& & L_1(v_1, u_2) L_2(u_1, v_2) & \xrightarrow{R^2(v_1, u_2|v_2)} & L_1(v_1, v_2) L_2(u_1, u_2)
\end{array}$$

which present two equivalent expressions for R-operator (2.12).

At certain parameter values the operators become simpler. Indeed, when  $u_1 = v_1$  or  $u_2 = v_2$  the corresponding permutation is trivial, consequently

$$R^1(u_1|u_1, v_2) = \mathbb{1}, \quad R^2(u_1, u_2|u_2) = \mathbb{1} \quad (2.13)$$

and

$$R(u_1, u_2|u_1, v_2) = R^2(u_1, u_2|v_2), \quad R(u_1, u_2|v_1, u_2) = R^1(u_1|v_1, u_2).$$

Notice that the latter relations hold for generic parameters only and modify in the case of integer values of  $u_2 - u_1$  or  $v_2 - v_1$ .

It is well known that along with the fundamental Yang-Baxter relation (2.2) and the RLL-relation (2.5) there is a further relation involving three general operators R. It can be understood as the formulation of the equivalence of two ways of transforming  $L_1(u_1, u_2)L_2(v_1, v_2)L_3(w_1, w_2) \rightarrow L_1(w_1, w_2)L_2(v_1, v_2)L_3(u_1, u_2)$

$$\begin{aligned}
R_{12}(v_1, v_2|w_1, w_2)R_{23}(u_1, u_2|w_1, w_2)R_{12}(u_1, u_2|v_1, v_2) &= \\
= R_{23}(u_1, u_2|v_1, v_2)R_{12}(u_1, u_2|w_1, w_2)R_{23}(v_1, v_2|w_1, w_2) &
\end{aligned} \quad (2.14)$$

This relation can be represented by the commutative diagram

$$\begin{array}{ccc}
L_1(v_1, v_2) L_2(u_1, u_2) L_3(w_1, w_2) & \xrightarrow{R_{23}(u_1, u_2|w_1, w_2)} & L_1(v_1, v_2) L_2(w_1, w_2) L_3(u_1, u_2) \\
\uparrow R_{12}(u_1, u_2|v_1, v_2) & & \downarrow R_{12}(v_1, v_2|w_1, w_2) \\
L_1(u_1, u_2) L_2(v_1, v_2) L_3(w_1, w_2) & & L_1(w_1, w_2) L_2(v_1, v_2) L_3(u_1, u_2) \\
\downarrow R_{23}(v_1, v_2|w_1, w_2) & & \uparrow R_{23}(u_1, u_2|v_1, v_2) \\
L_1(u_1, u_2) L_2(w_1, w_2) L_3(v_1, v_2) & \xrightarrow{R_{12}(u_1, u_2|w_1, w_2)} & L_1(w_1, w_2) L_2(u_1, u_2) L_3(v_1, v_2)
\end{array}$$



A further relation involves two R-operators and one operator  $R^2$ . In analogy it can be understood as the formulation of the equivalence of the two ways transforming  $L_1(u_1, u_2)L_2(v_1, v_2)L_3(w_1, w_2) \rightarrow L_1(v_1, w_2)L_2(w_1, v_2)L_3(u_1, u_2)$

$$\begin{aligned} R_{12}^2(v_1, v_2|w_2)R_{23}(u_1, u_2|w_1, w_2)R_{12}(u_1, u_2|v_1, v_2) = \\ = R_{23}(u_1, u_2|w_1, v_2)R_{12}(u_1, u_2|v_1, w_2)R_{23}^2(v_1, v_2|w_2) \end{aligned} \quad (2.15)$$

depicted by the diagram

$$\begin{array}{ccc} L_1(v_1, v_2) L_2(u_1, u_2) L_3(w_1, w_2) & \xrightarrow{R_{23}(u_1, u_2|w_1, w_2)} & L_1(v_1, v_2) L_2(w_1, w_2) L_3(u_1, u_2) \\ \uparrow R_{12}(u_1, u_2|v_1, v_2) & & \downarrow R_{12}^2(v_1, v_2|w_2) \\ L_1(u_1, u_2) L_2(v_1, v_2) L_3(w_1, w_2) & & L_1(v_1, w_2) L_2(w_1, v_2) L_3(u_1, u_2) \\ \downarrow R_{23}^2(v_1, v_2|w_2) & & \uparrow R_{23}(u_1, u_2|w_1, v_2) \\ L_1(u_1, u_2) L_2(v_1, w_2) L_3(w_1, v_2) & \xrightarrow{R_{12}(u_1, u_2|v_1, w_2)} & L_1(v_1, w_2) L_2(u_1, u_2) L_3(w_1, v_2) \end{array}$$

We have a third relation where instead of  $R^2$  the other factor operator  $R^1$  is involved.

$$\begin{aligned} R_{12}^1(v_1|w_1, w_2)R_{23}(u_1, u_2|w_1, w_2)R_{12}(u_1, u_2|v_1, v_2) = \\ = R_{23}(u_1, u_2|v_1, w_2)R_{12}(u_1, u_2|w_1, v_2)R_{23}^1(v_1|w_1, w_2). \end{aligned} \quad (2.16)$$

All these relations can be proved directly by using explicit expressions for all operators (2.11, 2.12) but, as V. Tarasov explained us, the very existence of operators  $R^1$  and  $R^2$  and all relations among them can be extracted from the general theory developed in papers [29].

Note that the general Yang-Baxter equation (2.14) is the consequence of (2.15) and (2.16) but the two latter are not a straightforward consequence of the general Yang-Baxter equation. Furthermore for our purpose they are as important as the general Yang-Baxter equation. Indeed, we will show that starting just only from these three relations (2.14), (2.15), (2.16) it is possible to deduce factorization of the general transfer matrix  $T(u)$  into the product of Baxter Q-operators, commutativity of all these operators and also to obtain the Baxter relation. No further constructions or concepts are needed.

### 3 A Q-operator constructed from $R^2$

In this section we would like to show the introduced operators at work. For this purpose we construct a Q-operator using the operators  $R^2$  as building blocks. Later we shall present a more systematic and exhaustive construction. But now in this simple example we would like to show that the properties of a Q-operator follow directly from properties of its local building blocks.

#### 3.1 Transfer matrix $t(u)$

The closed homogeneous  $sl_2$ -symmetric spin chain under consideration consists of  $N$  sites carrying representations of the same representation parameter  $\ell$ . The operator  $L_k(u)$  (2.3) acts in the space  $\mathbb{U}_{-\ell} \otimes \mathbb{C}^2$ . The corresponding quantum space  $\mathbb{U}_{-\ell}$  is associated with each site and for generic  $\ell \in \mathbb{C}$  is isomorphic to the space  $\mathbb{C}[z_k]$  of polynomials of the variable  $z_k$ . The space  $\mathbb{C}^2$  is a common

auxiliary space. Then the product  $L_1(u) \cdots L_N(u)$  acts in the space  $\mathbb{U}_{-\ell} \otimes \cdots \otimes \mathbb{U}_{-\ell} \otimes \mathbb{C}^2$ . Taking trace over the auxiliary two-dimensional space  $\mathbb{C}^2$  we define the transfer matrix  $t(u)$

$$t(u) = \text{tr} L_1(u) L_2(u) \cdots L_N(u) \quad (3.1)$$

Recall the standard argument for commutativity of the ordinary transfer matrix,

$$[t(u), t(u)] = 0 \quad (3.2)$$

The fundamental Yang-Baxter relation (2.2) implies the analogous relation with the  $L$  matrices replaced by the monodromy matrix, i.e. the product of  $N$  matrices  $L_k$ , where the operator matrix elements act in  $\mathbb{U}_{-\ell} \sim \mathbb{C}[z_k]$ ,

$$L_{ij} \rightarrow L_{ik_1} L_{k_1 k_2} \cdots L_{k_{N-1} j} \quad (3.3)$$

Then taking traces in both 2-dimensional tensor factors results in the vanishing commutator of the transfer matrices. The proofs of factorization and commutativity for the other global spin chain operators to be defined below follow basically this scheme starting from the appropriate form of the Yang-Baxter relation.

Since  $t(u)$  is polynomial in the spectral parameter we obtain the family of  $N - 1$  commuting operators acting in the space  $\mathbb{U}_{-\ell} \otimes \cdots \otimes \mathbb{U}_{-\ell}$ . In the generic situation  $\ell \in \mathbb{C}$  this quantum space is isomorphic to the space  $\mathbb{C}[z_1, \cdots, z_N]$  of polynomials depending on variables  $z_1, \cdots, z_N$ . The reference state  $|0\rangle$  of ABA plays the common lowest weight vector for all representations  $\mathbb{U}_{-\ell}$ :  $S_k^- |0\rangle = 0$ , i.e. the polynomial of zero degree, the constant function. The eigenvalues of the operators  $A(u)$  and  $D(u)$  appearing in (1.1) can be easily calculated so that we have explicit expressions for the functions  $\Delta_{\pm}(u)$ :  $\Delta_{\pm}(u) = (u \mp \ell)^N$ .

### 3.2 Q-operator and Baxter equation

The left hand side of the Baxter equation (1.6) involves the product of the transfer matrix  $t(u)$  and the  $Q$ -operator. The former is constructed from  $L$ -operators and we are going to construct the latter from operators  $R^2$ . Thus we need a local relation which comprise the product of  $R^2$  and  $L$  operators. It becomes clear soon that the defining equation for the operator  $R^2$  (2.10) fits very well for this purpose.

Using the factorization formulae (2.4) for  $L_1(u_1, u_2)$  and  $L_2(v_1, v_2)$  and the commutativity  $R^2 z_2 = z_2 R^2$  we rewrite (2.10) in a slightly different form

$$Z_1^{-1} R_{12}^2(u-v_2) L_1(u_1, u_2) Z_2 = \begin{pmatrix} u_1 & -\partial_1 \\ 0 & v_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -z_{12} & 1 \end{pmatrix} \begin{pmatrix} v_1 & -\partial_2 \\ 0 & u_2 \end{pmatrix} \cdot R_{12}^2(u-v_2) \cdot \begin{pmatrix} v_1 & -\partial_2 \\ 0 & v_2 \end{pmatrix}^{-1}.$$

We see that the dependence of  $R^2$  on the parameter  $v_2$  enters by a shift of the spectral parameter  $u$ .  $Z_k$  are triangular matrices.

$$R_{12}^2(u - v_2) = \partial_1^{u_1 - v_2} z_{12}^{u_2 - v_2} \partial_1^{u_2 - u_1} \quad ; \quad Z_k \equiv \begin{pmatrix} 1 & 0 \\ z_k & 1 \end{pmatrix}.$$

Next we have to calculate the product of matrices at right hand side. For our purpose we need only the diagonal elements of the matrix. We obtain as upper diagonal element  $(u_1 + \partial_1 z_{12}) R_{12}^2(u - v_2)$  which is transformed to the needed form using the relation

$$\partial_1 z_{12} R_{12}^2(u - v_2) = R_{12}^2(u + 1 - v_2) - (u_1 - v_2) R_{12}^2(u - v_2).$$

The lower diagonal element,  $u_2 R_{12}^2(u - v_2) + z_{12} [\partial_2, R_{12}^2(u - v_2)]$ , is transformed to the final form by using the relation

$$z_{12} [\partial_2, R_{12}^2(u - v_2)] = (v_2 - u_2) R_{12}^2(u - v_2) + (u_1 - v_2)(u_2 - v_2) R_{12}^2(u - 1 - v_2)$$

Performing this calculation we use only the simple commutation properties of the operator (2.8). Finally we obtain

$$\begin{aligned} & Z_1^{-1} R_{12}^2(u - v_2) L_1(u_1, u_2) Z_2 = \\ & = \begin{pmatrix} R_{12}^2(u + 1 - v_2) + v_2 R_{12}^2(u - v_2) & -R_{12}^2(u - v_2) \partial_1 \\ -v_2 z_{12} R_{12}^2(u - v_2) & (u_1 - v_2)(u_2 - v_2) R_{12}^2(u - 1 - v_2) + v_2 R_{12}^2(u - v_2) \end{pmatrix}. \end{aligned}$$

The crucial feature of this relation being the cornerstone of the current calculation is that at the point  $v_2 = 0$  the matrix on right hand side becomes upper triangular. We put  $v_2 = 0$  in the derived matrix relation and specify it by choosing the first space to be the local quantum space at site  $k$  and the second space the local quantum space at site  $(k + 1)$

$$Z_k^{-1} \cdot R_{kk+1}^2(u) L_k(u_1, u_2) \cdot Z_{k+1} = \begin{pmatrix} R_{kk+1}^2(u + 1) & -R_{kk+1}^2(u) \partial_k \\ 0 & u_1 u_2 R_{kk+1}^2(u - 1) \end{pmatrix}.$$

This is exactly the local relation which underlies Baxter equation. We take the product over all sites

$$\begin{aligned} & Z_1^{-1} \cdot R_{12}^2(u) R_{23}^2(u) \cdots R_{N0}^2(u) \cdot L_1(u) L_2(u) \cdots L_N(u) \cdot Z_0 = \\ & = \begin{pmatrix} R_{12}^2(u + 1) & -R_{12}^2(u) \partial_1 \\ 0 & u_1 u_2 R_{12}^2(u - 1) \end{pmatrix} \cdots \begin{pmatrix} R_{N0}^2(u + 1) & -R_{N0}^2(u) \partial_N \\ 0 & u_1 u_2 R_{N0}^2(u - 1) \end{pmatrix} \end{aligned} \quad (3.4)$$

appending one additional fictitious site 0. In this product the matrices  $Z_k$  and  $Z_k^{-1}$  ( $k = 2, 3, \dots, N$ ) cancel pairwise. Then we calculate the trace over the two-dimensional space  $\mathbb{C}^2$ , use commutativity of all  $R^2$  and  $L_k$  with  $z_0$  in order to move  $Z_0$  to the left and finally identify sites 0 and 1. Thus we obtain the Baxter equation

$$t(u) Q_2(u) = Q_2(u + 1) + (u_1 u_2)^N \cdot Q_2(u - 1) \quad (3.5)$$

for the operator<sup>1</sup>

$$Q_2(u) = P \cdot R_{12}^2(u) R_{23}^2(u) \cdots R_{N-1,N}^2(u) R_{N0}^2(u) \Big|_{z_0 \rightarrow z_1}. \quad (3.6)$$

We have constructed explicitly a solution of the Baxter equation relying on the local relation (2.10) only. It is also possible to check in this way the commutativity properties of  $Q_2(u)$  but we would like to obtain this result in the framework of the general scheme to be considered further. Note that the Baxter equation (1.5) can be transformed to the canonical form by appropriate normalization of the operator  $Q_2(u)$  which will appear naturally in the next subsection.

### 3.3 Explicit formulae for the action of $Q_2$ on polynomials

Now we would like to visualize the constructed operator  $Q_2$ . In order to use the advantages of the generating function method we combine the basis vectors of the module  $\mathbb{U}_{-\ell}$  into

$$e^{xS^+} \cdot 1 = (1 - xz)^{2\ell}, \quad (3.7)$$

where  $x$  is an auxiliary parameter. The derivative  $\partial_x^k$  at the point  $x = 0$  produces the basis vector  $S_+^k \cdot 1 \sim z^k$ . We shall obtain a very simple representation for  $Q_2(u)$  acting on the global generating function  $(1 - x_1 z_1)^{2\ell} \cdots (1 - x_N z_N)^{2\ell}$ . This formula contains in transparent form all information about the action of the operator  $Q_2(u)$  on polynomials. Indeed, the calculation of the derivative  $\partial_{x_1}^{k_1} \cdots \partial_{x_N}^{k_N}$  at  $x_1 = \dots = x_N = 0$  results in the explicit expression for the action of the operator  $Q_2(u)$  on the monomial  $z_1^{k_1} \cdots z_N^{k_N}$ .

---

<sup>1</sup>We also multiply the last equation by cyclic shift  $P = P_{12} P_{13} \cdots P_{1N}$  for later convenience.

The special construction of the operator  $Q_2(u)$  reduces the global problem to a local one. Indeed the whole expression factorizes into the pieces of the simple form

$$\begin{aligned} R_{12}^2(u) R_{23}^2(u) \cdots R_{N0}^2(u) \cdot (1 - x_1 z_1)^{2\ell} (1 - x_2 z_2)^{2\ell} \cdots (1 - x_N z_N)^{2\ell} = \\ = R_{12}^2(u) (1 - x_1 z_1)^{2\ell} \cdot R_{23}^2(u) (1 - x_2 z_2)^{2\ell} \cdots R_{N0}^2(u) (1 - x_N z_N)^{2\ell} \end{aligned} \quad (3.8)$$

so that we have to calculate the local quantity  $R_{kk+1}^2(u) (1 - x_k z_k)^{2\ell}$ . It turns out that this expression can be obtained from "first symmetry principles" almost without calculations.

We start with some intertwining properties of  $R^2$ . Since it depends on the difference of the spectral parameters the shift  $u \rightarrow u + \lambda$ ,  $v \rightarrow v + \lambda$  does not affect  $R_{12}^2$ . Thus performing this shift in the defining relation (2.10) and collecting terms linear in  $\lambda$  we obtain

$$R_{12}^2 \cdot (L_1(u_1, u_2) + L_2(v_1, v_2)) = (L_1(u_1, v_2) + L_2(v_1, u_2)) \cdot R_{12}^2.$$

Since L-operators are constructed from generators of the symmetry algebra the previous relation implies

$$R_{12}^2 \cdot (S_1^+(\ell_1) + S_2^+(\ell_2)) = (S_1^+(\ell_1 - \alpha) + S_2^+(\ell_2 + \alpha)) \cdot R_{12}^2 \quad (3.9)$$

where we show explicitly the spin parameter of generators  $S_k^+(\ell) = z_k^2 \partial_k - 2\ell z_k$  and introduce the notation  $\alpha = \frac{u_2 - v_2}{2}$ . This relation and the commutativity of  $R_{12}^2$  with  $z_2$  allow to compute the action of  $R_{12}^2(u)$  on the generating function  $(1 - x z_1)^{2\ell_1}$ . The whole calculation is divided into three steps: first we transform the initial expression using commutativity  $R_{12}^2 z_2 = z_2 R_{12}^2$

$$R_{12}^2 (1 - x z_1)^{2\ell_1} = (1 - x z_2)^{-2\ell_2} \cdot R_{12}^2 \cdot (1 - x z_1)^{2\ell_1} (1 - x z_2)^{2\ell_2},$$

then use the representation for the generating function (3.7) and use also the intertwining relation (3.9)

$$R_{12}^2 \cdot \exp x (S_{\ell_1}^+ + S_{\ell_2}^+) \cdot 1 = \exp x (S_{\ell_1 - \alpha}^+ + S_{\ell_2 + \alpha}^+) \cdot R_{12}^2 \cdot 1,$$

calculate the emerging constant  $C = R_{12}^2 \cdot 1 = \frac{\Gamma(u_1 + 1)}{\Gamma(u_1 - u_2 + 1)}$  and use (3.7) once more

$$\exp x (S_{\ell_1 - \alpha}^+ + S_{\ell_2 + \alpha}^+) \cdot 1 = (1 - x z_1)^{2\ell_1 - 2\alpha} (1 - x z_2)^{2\ell_2 + 2\alpha}.$$

Collecting everything we arrive at

$$R_{12}^2 (1 - x z_1)^{2\ell_1} = C \cdot (1 - x z_1)^{2\ell_1 - 2\alpha} \cdot (1 - x z_2)^{2\alpha}.$$

Going back to (3.8) we fix  $v_2 = 0$ ;  $\ell_1 = \ell_2 = \ell$ , use the specification to arbitrary sites  $k, k + 1$

$$R_{kk+1}^2(u) (1 - x_k z_k)^{2\ell} = \frac{\Gamma(-\ell + u)}{\Gamma(-2\ell)} \cdot (1 - x_k z_k)^{\ell - u} \cdot (1 - x_k z_{k+1})^{\ell + u} \quad (3.10)$$

and obtain the closed expression for the action of the considered operator on the generating function

$$\begin{aligned} R_{12}^2(u) R_{23}^2(u) \cdots R_{N0}^2(u) \cdot (1 - x_1 z_1)^{2\ell} \cdots (1 - x_N z_N)^{2\ell} = \\ = \frac{\Gamma^N(-\ell + u)}{\Gamma^N(-2\ell)} \cdot (1 - x_1 z_1)^{\ell - u} (1 - x_1 z_2)^{\ell + u} \cdots (1 - x_N z_N)^{\ell - u} (1 - x_N z_0)^{\ell + u}. \end{aligned}$$

In order to obtain the wanted formula for the action of  $Q_2(u)$  on the generating function it remains to put  $z_0 = z_1$  and to perform the cyclic shift P.

The evident drawback of this formula is the presence of  $\Gamma$ -functions. To improve this we introduce the renormalized operator

$$Q(u) = \frac{\Gamma^N(-2\ell)}{\Gamma^N(-\ell+u)} \cdot Q_2(u) . \quad (3.11)$$

Its action on the generating function looks simpler

$$\begin{aligned} Q(u) : (1 - x_1 z_1)^{2\ell} \cdots (1 - x_N z_N)^{2\ell} &\mapsto \\ \mapsto (1 - x_1 z_N)^{\ell-u} (1 - x_1 z_1)^{\ell+u} \cdots (1 - x_N z_{N-1})^{\ell-u} (1 - x_N z_N)^{\ell+u} \end{aligned} \quad (3.12)$$

The operator  $Q(u)$  has two further advantages. It is normalized in a such way that  $Q(u) : 1 \mapsto 1$  and it is evident that it maps any monomial  $z_1^{k_1} \cdots z_N^{k_N}$  to polynomial with respect to variables  $z_1, \dots, z_N$  and the spectral parameter  $u$  so that it maps polynomials in  $z_1 \cdots z_N$  into polynomials in  $u, z_1 \cdots z_N$

$$Q(u) : \mathbb{C}[z_1 \cdots z_N] \mapsto \mathbb{C}[u, z_1 \cdots z_N] .$$

This property guarantees that the operator  $Q(u)$  has polynomial in  $u$  eigenvalues and that the polynomials  $Q_k(u)$  appearing in the algebraic Bethe ansatz approach (1.7) are just the eigenvalues of the  $Q$ -operator.

Finally it is easy to check that we obtain the canonical form of the Baxter equation for this improved operator

$$t(u)Q(u) = (u - \ell)^N \cdot Q(u + 1) + (u + \ell)^N \cdot Q(u - 1) .$$

Note that this  $Q(u)$  coincides explicitly with the  $Q$ -operator constructed by another method in [11].

## 4 Global objects: commuting transfer matrices

In section 2 we have introduced and investigated local operators which concern only one site of the spin chain. Now we turn to the description of the whole system. We are going to construct various generating functions of commuting operators, transfer matrices and Baxter  $Q$ -operators, from general  $\mathbb{R}$ -operators studied above.

### 4.1 General transfer matrix and factorization into $Q$ -operators

It is of interest to generalize the previous construction of the transfer matrix  $t(u)$ . The construction of the general transfer matrix  $T(u)$  substitutes in formula (3.1)  $L_k(u)$  as local operators by  $\mathbb{R}_{k0}(u|\ell, s)$  acting in the tensor product of quantum space  $\mathbb{U}_{-\ell}$  and auxiliary space  $\mathbb{U}_{-s}$ . The trace is taken over the generic infinite-dimensional auxiliary space

$$T_s(u) = \text{tr}_0 \mathbb{R}_{10}(u|\ell, s) \mathbb{R}_{20}(u|\ell, s) \cdots \mathbb{R}_{N0}(u|\ell, s) \quad (4.1)$$

At fixed spin  $\ell$  the free parameters in the general transfer matrix  $T_s(u)$  are the spectral parameter  $u$  and the spin parameter  $s$  in the auxiliary space. We recall the relation to our four-parameter notation (2.6),

$$\begin{aligned} \mathbb{R}_{k0}(u - v|\ell, s) &= \mathbb{R}_{k0}(u_1, u_2; v_1, v_2), \\ u_1 &= u - \ell - 1, u_2 = u + \ell, v_1 = v - s - 1, v_2 = v + s. \end{aligned}$$

In this notation the above definition can be rewritten as

$$T_s(u - v) = \text{tr}_0 \mathbb{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2). \quad (4.2)$$

The general transfer matrix has the remarkable factorization property

$$P \cdot T_s(u - v) = Q_2(u - v_2) Q_1(u - v_1) = Q_1(u - v_1) Q_2(u - v_2) \quad (4.3)$$

where operators  $Q_1$  and  $Q_2$  are transfer matrices constructed from operators  $\mathbb{R}_{k0}^1$  and  $\mathbb{R}_{k0}^2$

$$Q_1(u - v_1) = \text{tr}_0 \mathbb{R}_{10}^1(u_1|v_1, u_2) \cdots \mathbb{R}_{N0}^1(u_1|v_1, u_2), \quad (4.4)$$

$$Q_2(u - v_2) = \text{tr}_0 \mathbb{R}_{10}^2(u_1, u_2|v_2) \cdots \mathbb{R}_{N0}^2(u_1, u_2|v_2), \quad (4.5)$$

and the operator  $P = P_{12}P_{13} \cdots P_{1N}$  is the cyclic permutation along the closed chain.

Note that the dependence on parameters  $v_1$  and  $v_2$  results in a simple shift of spectral parameter,  $u \rightarrow u - v_1$  in the first operator  $Q_1$  and  $u \rightarrow u - v_2$  in the second one. Eliminating the redundant shift of spectral parameter ( $u - v \rightarrow u$ ) we have

$$P \cdot T_s(u) = Q_2(u - s) Q_1(u + s + 1) = Q_1(u + s + 1) Q_2(u - s).$$

The factorization (4.3) of transfer matrices generalizes the corresponding properties of its building blocks

$$\mathbb{R}_{12}(u_1, u_2|v_1, v_2) = P_{12} \mathbb{R}_{12}^1(u_1|v_1, u_2) \mathbb{R}_{12}^2(u_1, u_2|v_2) \quad (4.6)$$

The global factorization follows from the local three term relations (2.15) and (2.16). For doing the proof we start from the relation (2.15) and rewrite it for the operators with permutations included  $\mathbb{R}_{ik} = P_{ik} \mathbb{R}_{ik}$

$$\begin{aligned} \mathbb{R}_{23}^2(v_1, v_2|w_2) \mathbb{R}_{13}(u_1, u_2|w_1, w_2) \mathbb{R}_{12}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{12}(u_1, u_2|w_1, v_2) \mathbb{R}_{13}(u_1, u_2|v_1, w_2) \mathbb{R}_{23}^2(v_1, v_2|w_2). \end{aligned}$$

Now we choose the first space to be the local quantum space  $\mathbb{U}_{-\ell}$  in site  $k$ , the second space to be the auxiliary space  $\mathbb{U}_{-s} \sim \mathbb{C}[z_0]$  and the third space to be a second copy of the auxiliary space  $\mathbb{U}_{-s} \sim \mathbb{C}[z_{0'}]$

$$\begin{aligned} \mathbb{R}_{00'}^2(v_1, v_2|w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{k0}(u_1, u_2|w_1, v_2) \mathbb{R}_{k0'}(u_1, u_2|v_1, w_2) \mathbb{R}_{00'}^2(v_1, v_2|w_2). \end{aligned} \quad (4.7)$$

Recall that  $\mathbb{R}$ -operators simplify when some of the their parameters coincide

$$\mathbb{R}(u_1, u_2|u_1, v_2) = \mathbb{R}^2(u_1, u_2|v_2), \quad \mathbb{R}(u_1, u_2|v_1, u_2) = \mathbb{R}^1(u_1|v_1, u_2),$$

so that specifying the parameters  $w_1$  and  $w_2$  as  $w_1 = u_1$  and  $w_2 = u_2$  we obtain the local intertwining relation with the operator  $\mathbb{R}_{00'}^2(v_1, v_2|u_2)$

$$\mathbb{R}_{00'}^2(v_1, v_2|u_2) \cdot P_{k0'} \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \cdot \mathbb{R}_{k0'}^1(u_1|v_1, u_2) \cdot \mathbb{R}_{00'}^2(v_1, v_2|u_2)$$

leading in the standard way to the relation for the transfer matrices

$$\begin{aligned} \text{tr}_{0'} [P_{10'} \cdots P_{N0'}] \cdot \text{tr}_0 [\mathbb{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2)] = \\ = \text{tr}_0 [\mathbb{R}_{10}^2(u_1, u_2|v_2) \cdots \mathbb{R}_{N0}^2(u_1, u_2|v_2)] \cdot \text{tr}_{0'} [\mathbb{R}_{10'}^1(u_1|v_1, u_2) \cdots \mathbb{R}_{N0'}^1(u_1|v_1, u_2)]. \end{aligned} \quad (4.8)$$

We see that the general transfer matrix constructed from operators  $\mathbb{R}_{k0}(u_1, u_2|v_1, v_2)$  factorizes into the product of two transfer matrices constructed from  $\mathbb{R}_{k0}^2(u_1, u_2|v_2)$  and  $\mathbb{R}_{k0'}^1(u_1|v_1, u_2)$ . This factorization for global objects follows in a clear and direct way from the local relations for their building blocks. The similar proof of the second factorization is given in Appendix A.

The commutativity properties of the different general transfer matrices can be summarized as follows

$$[T_s(u), Q_k(v)] = 0 \quad ; \quad [Q_i(u), Q_k(v)] = 0 \quad ; \quad [P, Q_k(u)] = 0 \quad ; \quad i, k = 1, 2.$$

The direct consequence of the two factorizations (4.8) and (A.2) is the commutativity of the transfer matrices  $Q_1$  and  $Q_2$ . However it is more instructive to derive commutativity from local intertwining relations. In Appendix A for completeness we list the necessary relations.

Summarizing we have deduced factorization and commutativity properties for the transfer matrices with infinite-dimensional auxiliary spaces starting only from local relations (2.14), (2.16), (2.15).

The transfer matrices  $Q_1$  and  $Q_2$  constructed from operators  $\mathbb{R}^1$  and  $\mathbb{R}^2$  have all properties of the Q-operators, introduced by R.Baxter [5] and the second one coincides with the operator  $Q_2(u)$  constructed previously "by hand". For this reason we denote these transfer operators by  $Q_k$ .

From the list of the defining properties for Q-operators given in Introduction all commutativity properties are proven already so that we shall focus in the following on the Baxter equation.

In section 3 we have already constructed  $Q_2$  (3.6) "by hand" as particular solution of the Baxter equation. In Appendix B it is shown that the trace over infinite-dimensional space in (4.5) can be calculated explicitly due to specific form of  $R^2$  and it produces exactly (3.6), i.e. the two operators coincide. Thus we have an explicit useful formulae for the action of  $Q_2$  on polynomials. Using the slightly modified argumentation of section 3 it is easy to derive Baxter equation for  $Q_2$  in the form (4.5) directly from the defining relation of the operator  $R^2$  and the same is true for  $Q_1$  in the form (4.4).

For completeness we present another derivation [25, 27] of the Baxter equation which is deeper and shows the origin of this equation: at integer values of  $2s$  there appears a finite-dimensional invariant subspace inside the infinite-dimensional representation of the algebra  $sl_2$  in auxiliary space and this finally results in the Baxter equation. The general transfer matrix is constructed from general  $\mathbb{R}$ -operators so that we have in principle to consider at first the restriction to the finite-dimensional invariant subspaces of these local building blocks. All this will be discussed in detail in the next section. Here we simply use the needed formulae postponing their proof to the more appropriate moment.

## 4.2 Baxter equation and determinant formula

In the previous section we have introduced the general transfer matrix (4.1) for generic spin parameter  $s$  in the auxiliary space  $\mathbb{U}_{-s}$ . Now we are going to chose this parameter to be  $s = \frac{n}{2}$ ,  $n = 0, 1, 2, \dots$ . As we have mentioned in 2.1 in this case the auxiliary module  $\mathbb{U}_{-\frac{n}{2}}$  is now reducible. It is useful to the introduce operator  $\mathcal{D} = \partial^{n+1}$  intertwining generators of the algebra with parameters  $\frac{n}{2}$  and  $-\frac{n}{2} - 1$

$$\mathcal{D} \cdot S_{\pm} \left( \frac{n}{2} \right) = S_{\pm} \left( -1 - \frac{n}{2} \right) \cdot \mathcal{D} \quad ; \quad \mathcal{D} \cdot S \left( \frac{n}{2} \right) = S \left( -1 - \frac{n}{2} \right) \cdot \mathcal{D} . \quad (4.9)$$

It is easy to see that image and kernel of the operator  $\mathcal{D}$  are invariant subspaces. The kernel is the  $(n+1)$ -dimensional space  $\mathbb{V}_n$  with the basis  $\{1, z, \dots, z^n\}$  and the image is the infinite-dimensional space with basis  $\{1, z, z^2, \dots\}$  which is also irreducible. The operator  $\mathcal{D}$  maps the reducible module with lowest weight  $-\frac{n}{2}$  into the irreducible module with lowest weight  $\frac{n}{2} + 1$ :  $\text{Im} \mathcal{D} = \mathbb{U}_{\frac{n}{2}+1}$ ,  $\text{Ker} \mathcal{D} = \mathbb{V}_n$

$$\mathbb{U}_{-\frac{n}{2}} \xrightarrow{\mathcal{D}} \mathbb{U}_{\frac{n}{2}+1} \quad ; \quad \mathbb{V}_n \xrightarrow{\mathcal{D}} 0,$$

where the irreducible module  $\mathbb{U}_{\frac{n}{2}+1}$  is isomorphic to the quotient module  $\mathbb{U}_{\frac{n}{2}+1} \approx \mathbb{U}_{-\frac{n}{2}} / \mathbb{V}_n$  and the isomorphism is induced by intertwining operator  $\mathcal{D}$ . As the consequence the trace over  $\mathbb{U}_{-\frac{n}{2}}$  splits into traces over finite-dimensional  $\mathbb{V}_n$  and infinite-dimensional  $\mathbb{U}_{\frac{n}{2}+1}$ .

Applying this statement to the general transfer matrix we obtain its decomposition into the transfer matrix with finite-dimensional auxiliary space and the general transfer matrix with the other spin parameter:

$$T_{\frac{n}{2}}(u) = t_n(u) + T_{-\frac{n}{2}-1}(u). \quad (4.10)$$

Here in  $t_n(u)$  the trace is taken over  $\mathbb{V}_n$  and it represents the generalization of the ordinary transfer matrix  $t(u)$  considered in the section 3.1.

$$t_n(u) = \text{tr } \mathbf{R}_{10} \left( u | \ell, \frac{n}{2} \right) \mathbf{R}_{20} \left( u | \ell, \frac{n}{2} \right) \dots \mathbf{R}_{N0} \left( u | \ell, \frac{n}{2} \right). \quad (4.11)$$

$\mathbf{R}_{k0}(u | \ell, \frac{n}{2})$  is the restriction of the general  $\mathbb{R}$ -operator to the space  $\mathbb{U}_{-\ell} \otimes \mathbb{V}_n$ . In section 5.1 we shall calculate explicitly such restrictions for one- and two-dimensional auxiliary spaces. Using formula (5.3) we have <sup>2</sup>

$$t_0(u) = (-1)^{N(u-\ell)} \frac{\Gamma^N(-\ell+u)}{\Gamma^N(-\ell-u)} \cdot \mathbb{1}, \quad (4.12)$$

and (5.4) allows to connect  $t_1(u)$  with the standard transfer matrix  $t(u)$  considered in the section 3.1

$$t_1 \left( u - \frac{1}{2} \right) = (-1)^{N(u-\ell)} \frac{\Gamma^N(-\ell+u)}{\Gamma^N(-\ell-u)} \cdot \frac{t(u)}{(u_1 u_2)^N}.$$

The trace in  $T_{-\frac{n}{2}-1}(u)$  is taken over  $\mathbb{U}_{\frac{n}{2}+1}$ . The operator  $\mathcal{D}$  is invertible on this space and using the invariance of the trace with respect to similarity transformation we can write

$$T_{-\frac{n}{2}-1}(u) = \text{tr } \mathcal{D} \mathbb{R}_{10} \left( u | \ell, \frac{n}{2} \right) \mathbb{R}_{20} \left( u | \ell, \frac{n}{2} \right) \dots \mathbb{R}_{N0} \left( u | \ell, \frac{n}{2} \right) \mathcal{D}^{-1}.$$

Using the similarity transformation in the RLL-relation (2.5) and the intertwining property (4.9) of the operator  $\mathcal{D}$  one obtains

$$T_{-\frac{n}{2}-1}(u) = \text{tr } \mathbb{R}_{10} \left( u | \ell, -\frac{n}{2} - 1 \right) \mathbb{R}_{20} \left( u | \ell, -\frac{n}{2} - 1 \right) \dots \mathbb{R}_{N0} \left( u | \ell, -\frac{n}{2} - 1 \right)$$

that is in agreement with definition of (4.1).

The factorization properties of the general transfer matrix (4.3) lead then from (4.10) to the determinant formula

$$\begin{aligned} P \cdot t_n(u) &= Q_1 \left( u + \frac{n}{2} + 1 \right) Q_2 \left( u - \frac{n}{2} \right) - Q_1 \left( u - \frac{n}{2} \right) Q_2 \left( u + \frac{n}{2} + 1 \right) = \\ &= \begin{vmatrix} Q_1 \left( u + \frac{n}{2} + 1 \right) & Q_2 \left( u + \frac{n}{2} + 1 \right) \\ Q_1 \left( u - \frac{n}{2} \right) & Q_2 \left( u - \frac{n}{2} \right) \end{vmatrix} \end{aligned}$$

allowing to express the transfer matrix with finite-dimensional auxiliary space in terms of Baxter Q-operators. Then we proceed to establish a set of relations which are linear in transfer matrices and Q-operators [25]. Let us consider the following determinant which is zero due to equality of two columns

$$\begin{vmatrix} Q_1(a) & Q_2(a) & Q_k(a) \\ Q_1(b) & Q_2(b) & Q_k(b) \\ Q_1(c) & Q_2(c) & Q_k(c) \end{vmatrix} = 0 \quad ; \quad k = 1, 2.$$

---

<sup>2</sup> In this subsection we change normalization of R-operators  $R(u_1, u_2 | v_1, v_2) \rightarrow (-1)^{u_1 - v_1} R(u_1, u_2 | v_1, v_2)$  and  $R^1(u_1 | v_1, v_2) \rightarrow (-1)^{u_1 - v_1} R^1(u_1 | v_1, v_2)$  in order to obtain Baxter relation in the standard form. In the other parts of the paper we do not retain this normalization factor since it can be restored easily.



The decomposition with respect to the third column results in

$$\begin{vmatrix} Q_1(b) & Q_2(b) \\ Q_1(c) & Q_2(c) \end{vmatrix} \cdot Q_k(a) - \begin{vmatrix} Q_1(a) & Q_2(a) \\ Q_1(c) & Q_2(c) \end{vmatrix} \cdot Q_k(b) + \begin{vmatrix} Q_1(a) & Q_2(a) \\ Q_1(b) & Q_2(b) \end{vmatrix} \cdot Q_k(c) = 0.$$

Specifying parameters

$$a = u + \frac{n}{2} + 1 ; \quad b = u - \frac{n}{2} ; \quad c = u - m - \frac{n}{2} - 1$$

we see that the quadratic determinants turns into transfer matrices and we get the set of relations at  $n, m = 0, 1, 2, \dots$

$$\begin{aligned} t_m \left( u - 1 - \frac{n+m}{2} \right) \cdot Q_k \left( u + 1 + \frac{n}{2} \right) - t_{n+m+1} \left( u - \frac{m+1}{2} \right) \cdot Q_k \left( u - \frac{n}{2} \right) + \\ + t_n(u) \cdot Q_k \left( u - 1 - m - \frac{n}{2} \right) = 0 \end{aligned}$$

Let us assign in the previous relation  $n = m = 0$

$$t_1 \left( u - \frac{1}{2} \right) \cdot Q_k(u) = t_0(u-1) \cdot Q_k(u+1) + t_0(u) \cdot Q_k(u-1). \quad (4.13)$$

Taking into account the expressions for transfer matrices with 1-dimensional and 2-dimensional auxiliary spaces (4.11), (4.12) we obtain Baxter relations

$$t(u) \cdot Q_k(u) = Q_k(u+1) + (u_1 u_2)^N \cdot Q_k(u-1).$$

### 4.3 Explicit formulae for action on polynomials

Above we have given the mostly algebraic construction of Q-operators acting in infinite-dimensional quantum space in the case of generic spin  $\ell \in \mathbb{C}$ . Now we are going to consider explicit formulae for the action of these operators on polynomials.

We start from the uniform expressions for the operators  $Q_1$  and  $Q_2$

$$Q_1(u) = \text{tr}_{V_0} P_{10} R_1(z_{01} \partial_0) \cdots P_{N0} R_1(z_{N0} \partial_0),$$

$$Q_2(u) = \text{tr}_{V_0} P_{10} R_2(z_{10} \partial_1) \cdots P_{N0} R_2(z_{N0} \partial_N),$$

where we use the following functions of its operator arguments

$$R_1(x) = \frac{\Gamma(x-2\ell)}{\Gamma(x+1-\ell-u)} \quad ; \quad R_2(x) = \frac{\Gamma(x+u-\ell)}{\Gamma(x-2\ell)}. \quad (4.14)$$

This expression suggests the derivation of the explicit expression for the action on polynomials. We use a simple trick for separating the dependence on spectral parameter  $u$  and spin  $\ell$ . The evident formula

$$\Phi(\lambda \partial_\lambda)|_{\lambda=1} \cdot \lambda^x = \Phi(x)$$

allows to extract the dependence on  $u$  and  $\ell$  into operators involving  $\lambda \partial_\lambda$

$$Q_1(u) = R_1(\lambda_1 \partial_{\lambda_1}) \cdots R_1(\lambda_N \partial_{\lambda_N})|_{\lambda=1} \cdot \text{tr}_{V_0} P_{10} \lambda_1^{z_{01} \partial_0} \cdots P_{N0} \lambda_N^{z_{N0} \partial_0},$$

$$Q_2(u) = R_2(\lambda_1 \partial_{\lambda_1}) \cdots R_2(\lambda_N \partial_{\lambda_N})|_{\lambda=1} \cdot \text{tr}_{V_0} P_{10} \lambda_1^{z_{10} \partial_1} \cdots P_{N0} \lambda_N^{z_{N0} \partial_N}.$$

After this transformation we can focus on the problem of calculation of the traces in these two basic expressions. For the calculation of the trace in the space of polynomials  $\mathbb{C}[z_0]$  we adopt the standard procedure. We use the monomial basis  $e_k = z_0^k$  in the space  $\mathbb{C}[z_0]$  and the standard definition for

matrix of an operator  $Ae_i = \sum_k e_k A_{ki}$  and then calculate the trace of operator  $A$  as the sum of diagonal matrix elements  $\text{tr} A = \sum_k A_{kk}$ .

Let us start with the second operator because this trace is the simplest one. It is the special case of the general situation considered in Appendix B and the result of calculation of the trace is

$$\text{tr}_{V_0} P_{10} \lambda_1^{z_{10}\partial_1} \dots P_{N0} \lambda_N^{z_{N0}\partial_N} = P \cdot \lambda_1^{z_{12}\partial_1} \lambda_2^{z_{23}\partial_2} \dots \lambda_N^{z_{N0}\partial_N} \Big|_{z_0=z_1} \quad (4.15)$$

The result of the action of the obtained operator on a function  $\Psi(z_1, \dots, z_N) = \Psi(\vec{z})$  can be expressed in a compact matrix form

$$\lambda_1^{z_{12}\partial_1} \lambda_2^{z_{23}\partial_2} \dots \lambda_N^{z_{N0}\partial_N} \Big|_{z_0=z_1} \cdot \Psi(\vec{z}) = \Psi(\Lambda \vec{z}),$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 1 - \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 1 - \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 1 - \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_{N-1} & 1 - \lambda_{N-1} \\ 1 - \lambda_N & 0 & 0 & 0 & \dots & 0 & \lambda_N \end{pmatrix}; \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \dots \\ \dots \\ z_N \end{pmatrix}$$

Now we turn to the first operator  $Q_1$ . The calculation of the trace is based on the formula

$$\sum_{k=0}^{\infty} \frac{1}{k!} \partial_0^k \cdot (a + b \cdot z_0)^k \Phi(z_0) \Big|_{z_0=0} = \frac{1}{1-b} \cdot \Phi\left(\frac{a}{1-b}\right), \quad (4.16)$$

which is proven in Appendix B. Also the calculation of the involved trace  $\text{tr}_{V_0} P_{10} \lambda_1^{z_{01}\partial_0} \dots P_{N0} \lambda_N^{z_{0N}\partial_0}$  is given in Appendix B with the result

$$\text{tr}_{V_0} P_{10} \lambda_1^{z_{01}\partial_0} \dots P_{N0} \lambda_N^{z_{0N}\partial_0} \cdot \Psi(\vec{z}) = \frac{1}{1 - \bar{\lambda}_1 \dots \bar{\lambda}_N} \cdot \Psi(\Lambda'^{-1} \vec{z}), \quad (4.17)$$

where

$$\Lambda' = \begin{pmatrix} 1 - \frac{1}{\bar{\lambda}_1} & \frac{1}{\bar{\lambda}_1} & 0 & 0 & \dots & 0 \\ 0 & 1 - \frac{1}{\bar{\lambda}_2} & \frac{1}{\bar{\lambda}_2} & 0 & \dots & 0 \\ 0 & 0 & 1 - \frac{1}{\bar{\lambda}_3} & \frac{1}{\bar{\lambda}_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & 1 - \frac{1}{\bar{\lambda}_{N-1}} & \frac{1}{\bar{\lambda}_{N-1}} \\ \frac{1}{\bar{\lambda}_N} & 0 & 0 & 0 & \dots & 0 & 1 - \frac{1}{\bar{\lambda}_N} \end{pmatrix}.$$

As the result we obtain the very similar formulae for the action of the operators  $Q_k$  on polynomials

$$Q_1(u) \Psi(\vec{z}) = R_1(\lambda_1 \partial_{\lambda_1}) \dots R_1(\lambda_N \partial_{\lambda_N}) \Big|_{\lambda=1} \cdot \frac{1}{1 - \bar{\lambda}_1 \dots \bar{\lambda}_N} \cdot \Psi(\Lambda'^{-1} \vec{z}), \quad (4.18)$$

$$Q_2(u) \Psi(\vec{z}) = P \cdot R_2(\lambda_1 \partial_{\lambda_1}) \dots R_2(\lambda_N \partial_{\lambda_N}) \Big|_{\lambda=1} \cdot \Psi(\Lambda \vec{z}). \quad (4.19)$$

The main difference is in the prefactor and the inversion of the matrix in the first formula. To avoid misunderstanding we quote how the operator  $P$  acts on the function:  $P\Psi(z_1, z_2, \dots, z_N) = \Psi(z_N, z_1, \dots, z_{N-1})$ .

The formulae (4.18) and (4.19) are the starting points for the derivation of various representations for the  $Q$ -operators. One kind of representations is in the form of integral operator, which is obtained by using the simple substitutions

$$R_1(\lambda_k \partial_{\lambda_k}) \Big|_{\lambda_k=1} \rightarrow \frac{1}{\Gamma(1 + \ell - u)} \cdot \int_0^1 d\lambda_k (1 - \lambda_k)^{\ell-u} \lambda_k^{-2\ell-1}, \quad (4.20)$$

$$\mathbb{R}_2(\lambda_k \partial_{\lambda_k})|_{\lambda_k=1} \rightarrow \frac{1}{\Gamma(-\ell-u)} \cdot \int_0^1 d\lambda_k (1-\lambda_k)^{-\ell-u-1} \lambda_k^{-\ell+u-1} \quad (4.21)$$

so that we obtain the multiple integral representation for the operator  $\mathbb{Q}_1$

$$\begin{aligned} [\mathbb{Q}_1(u)\Psi](\vec{z}) &= \frac{1}{\Gamma^N(1+\ell-u)} \cdot \int_0^1 d\lambda_1 (1-\lambda_1)^{\ell-u} \lambda_1^{-2\ell-1} \dots \\ &\dots \int_0^1 d\lambda_N (1-\lambda_N)^{\ell-u} \lambda_N^{-2\ell-1} \frac{1}{1-\bar{\lambda}_1 \dots \bar{\lambda}_N} \cdot \Psi(\Lambda^{-1} \vec{z}), \end{aligned} \quad (4.22)$$

and for the operator  $\mathbb{Q}_2$  this leads to the multiple integral representation

$$\begin{aligned} [\mathbb{Q}_2(u)\Psi](z_1, \dots, z_N) &= \frac{1}{\Gamma^N(-\ell-u)} \cdot \int_0^1 d\lambda_1 (1-\lambda_1)^{-\ell-u-1} \lambda_1^{-\ell+u-1} \dots \\ &\dots \int_0^1 d\lambda_N (1-\lambda_N)^{-\ell-u-1} \lambda_N^{-\ell+u-1} \Psi(\lambda_1 z_N + \bar{\lambda}_1 z_1, \lambda_2 z_1 + \bar{\lambda}_2 z_2, \dots, \lambda_N z_{N-1} + \bar{\lambda}_N z_N). \end{aligned} \quad (4.23)$$

In fact the derivation of the rules (4.20) and (4.21) is reduced to the use of the beta-integral representation for corresponding operator: in the case  $\mathbb{R}_2(\lambda \partial_\lambda)$  it is the following chain of transformations

$$\begin{aligned} \mathbb{R}_2(\lambda \partial_\lambda) \Phi(\lambda)|_{\lambda=1} &= \frac{1}{\Gamma(-\ell-u)} \cdot \int_0^1 d\alpha (1-\alpha)^{-\ell-u-1} \alpha^{-\ell+u-1} \cdot \alpha^{\lambda \partial_\lambda} \Phi(\lambda)|_{\lambda=1} = \\ &= \frac{1}{\Gamma(-\ell-u)} \cdot \int_0^1 d\alpha (1-\alpha)^{-\ell-u-1} \alpha^{-\ell+u-1} \Phi(\alpha), \end{aligned}$$

and on the last step for simplicity in (4.21) we change  $\alpha \rightarrow \lambda$  again.

We postpone to the next section the derivation of the useful representation for the operator  $\mathbb{Q}_1$  in the case of half-integer spin. Here we add some comments about the operator  $\mathbb{Q}_2$  concerning the connection between the present formula (4.23) and representation (3.12) from section 3. For the generating function  $(1-x_1 z_1)^{2\ell} \dots (1-x_N z_N)^{2\ell}$  in left hand side of formula (3.12) the multiple integral (4.23) is factorized to the product of simple integrals calculated explicitly by using Feynman formula

$$\int_0^1 d\alpha \alpha^{a-1} (1-\alpha)^{b-1} \frac{1}{[\alpha A + (1-\alpha)B]^{a+b}} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \cdot \frac{1}{A^a B^b} \quad (4.24)$$

so that finally we arrive up to normalization at the right hand side of (3.12).

## 5 Finite-dimensional representations

In the previous section we have assumed the spin parameter  $\ell$  to be a generic complex number and consequently the quantum space of the model was infinite-dimensional. Under this assumptions we have concentrated on the algebraic properties of  $s\ell_2$ -invariant transfer matrices constructed from the operators  $\mathbb{R}^1$  and  $\mathbb{R}^2$  and have demonstrated the key properties that allow to call them Q-operators: commutativity and Baxter equation. Now we are going to consider the special situation of a half-integer spin  $\ell$ . In this case the infinite-dimensional representation becomes reducible and there an invariant subspace appears which is the standard finite-dimensional irreducible representation labeled by half-integer spin  $\ell$ .

What happens if we put in the formulae obtained so far  $\ell$  to be half-integer and restrict all our operators to the finite-dimensional quantum subspace? Let us turn to the formula (3.12) since it

is very useful for the discussion of the finite-dimensional representations. We consider the simplest example of spin  $\ell = \frac{1}{2}$  and two sites ( $N = 2$ ) for illustration.

$$Q(u) : (1 - x_1 z_1) \cdot (1 - x_2 z_2) \mapsto (1 - x_1 z_2)^{\frac{1}{2}-u} (1 - x_1 z_1)^{\frac{1}{2}+u} \cdot (1 - x_2 z_1)^{\frac{1}{2}-u} (1 - x_2 z_2)^{\frac{1}{2}+u}. \quad (5.1)$$

The tensor product basis from two quantum spaces with bases  $\{1, z_1\}$  and  $\{1, z_2\}$  is  $\{1, z_1, z_2, z_1 z_2\}$ . From formula (5.1) one easily obtains the action of the operator  $Q(u)$  on these basis vectors

$$\begin{aligned} z_1 &\mapsto \left(\frac{1}{2} + u\right) z_1 + \left(\frac{1}{2} - u\right) z_2 \quad ; \quad z_2 \mapsto \left(\frac{1}{2} + u\right) z_2 + \left(\frac{1}{2} - u\right) z_1 \\ 1 &\mapsto 1 \quad ; \quad z_1 z_2 \mapsto \left(\frac{1}{2} + 2u^2\right) z_1 z_2 + \left(\frac{1}{4} - u^2\right) (z_1^2 + z_2^2). \end{aligned}$$

Due to the presence of the term  $\sim (z_1^2 + z_2^2)$  the operator  $Q(u)$  maps beyond the initial four-dimensional space. It is clear that the same phenomenon occurs for any compact spin  $\ell = \frac{n}{2}$ ;  $n+1 \in \mathbb{N}$ .

We see that the operator  $Q_2$  maps beyond the quantum space and therefore we have to reconsider our construction in the case of finite-dimensional representations. Our strategy will be the following:

- From the very beginning we shall work with the restriction of the general  $\mathbb{R}$ -operator to the invariant subspace appearing for half-integer value of the spin.
- We use this restricted  $\mathbb{R}$ -operator as building block for the construction of the corresponding transfer matrix. At this stage we must introduce a regularization because of divergence of the trace over the infinite-dimensional space. In the case of arbitrary spin  $\ell \in \mathbb{C}$  the spin parameter  $\ell$  itself plays the role of regulator but now we fix it to be half-integer so that one needs a new regularization.
- Using local relations we prove factorization of the general transfer matrix into the product of the corresponding  $Q$ -operators.

The main difference to the case of generic  $\ell \in \mathbb{C}$  is that from very beginning all operators are restricted to invariant finite-dimensional subspace. Contrary to  $\mathbb{R}^1$  and  $\mathbb{R}^2$ , they map the finite-dimensional quantum space into itself. To start with we consider in details two examples of such restriction in the next subsection.

### 5.1 Examples of restriction: one- and two-dimensional representations

Now we are going to fill the gap left in the previous section and consider two particular examples of restriction needed for the derivation of the Baxter equation in the previous section 4.2. The  $\mathbb{R}$ -operator acts in the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  and according to (2.12) has the form

$$\mathbb{R}(u|\ell_1, \ell_2) = P_{12} \cdot \frac{\Gamma(z_{12}\partial_1 - 2\ell_2)}{\Gamma(z_{12}\partial_2 - \ell_1 - \ell_2 - u)} \cdot \frac{\Gamma(z_{21}\partial_2 - \ell_1 - \ell_2 + u)}{\Gamma(z_{21}\partial_2 - 2\ell_2)}. \quad (5.2)$$

Now we chose  $\ell_2 = \frac{n}{2}$  and restrict the  $\mathbb{R}$ -operator to the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{V}_n$ .

In this section we perform explicit and detailed calculations for  $n = 0, 1$ . In the case  $\ell_2 = 0$  we have

$$\mathbb{U}_{-\ell_1} \otimes \mathbb{V}_0 \sim \mathbb{U}_{-\ell_1} \otimes \mathbb{C}$$

so that the restricted  $\mathbb{R}$ -operator acts on functions of the form  $\Psi(z_1, z_2) = \phi(z_1)$ . The action of the first operator in (5.2) is simple due to independence on the variable  $z_2$

$$e^{-z_1\partial_2} \frac{\Gamma(z_2\partial_2 - \ell_1 - \ell_2 + u)}{\Gamma(z_2\partial_2 - 2\ell_2)} e^{z_1\partial_2} \cdot \phi(z_1) = \phi(z_1) \cdot \frac{\Gamma(-\ell_1 - \ell_2 + u)}{\Gamma(-2\ell_2)}.$$

Note that in the point  $\ell_2 = 0$  the above expression turns into zero. Therefore we have to introduce the regularization  $2\ell_2 = -\varepsilon$ . Then this expression has a simple zero at  $\varepsilon \rightarrow 0$

$$\phi(z_1) \cdot \frac{\Gamma(-\ell_1 + u)}{\Gamma(\varepsilon)} + O(\varepsilon^2).$$

The introduction of regularization implies that we do not substitute directly in (5.2) the value of compact spin but do perform carefully the limiting procedure. The action of the second operator is not trivial

$$e^{-z_2 \partial_1} \frac{\Gamma(z_1 \partial_1 - 2\ell_2)}{\Gamma(z_1 \partial_1 - \ell_1 - \ell_2 - u)} e^{z_2 \partial_1} \cdot \phi(z_1) = \sum_{k=0}^{\infty} \frac{\phi^{(k)}(z_2)}{k!} \frac{\Gamma(k + \varepsilon)}{\Gamma(k - \ell_1 + \frac{\varepsilon}{2} - u)} \cdot z_{12}^k$$

but we have actually to extract the singular part  $\sim \varepsilon^{-1}$  only. In other words previous expression with needed accuracy is equal to

$$\phi(z_1) \cdot \frac{\Gamma(\varepsilon)}{\Gamma(-\ell_1 - u)} + O(\varepsilon^0).$$

Now we see that the simple pole cancels out the simple zero and we have for  $\ell_1 = \ell$  and  $\ell_2 \rightarrow 0$

$$\mathbb{R}(u|\ell, 0) \phi(z_1) = \frac{\Gamma(-\ell + u)}{\Gamma(-\ell - u)} \cdot \phi(z_1)$$

or in the operator form

$$\mathbb{R}(u|\ell, 0) |_{\mathbb{V}_0} = \frac{\Gamma(-\ell + u)}{\Gamma(-\ell - u)} \cdot \mathbf{1}. \quad (5.3)$$

We would like to mention that the presence of divergences in the considered operators is not a plague but rather an advantage allowing to simplify the calculation considerably.

In the case  $\ell_2 = \frac{1}{2}$  we have

$$\mathbb{U}_{-\ell_1} \otimes \mathbb{V}_1 \sim \mathbb{U}_{-\ell_1} \otimes \mathbb{C}^2$$

and the R-operator acts on the functions of the form

$$\Psi(z_1, z_2) = \phi(z_1) + z_2 \psi(z_1).$$

The action of the first operator in (5.2) is simple due to special  $z_2$ -dependence

$$[\phi(z_1) + z_1 \psi(z_1) + z_{12} \cdot \psi(z_1) \cdot (-\ell_1 - \frac{1}{2} + u)] \cdot \frac{\Gamma(-\ell_1 - \frac{1}{2} + u)}{\Gamma(-1 + \varepsilon)} + O(\varepsilon^2)$$

As in the previous calculation we have to introduce regularization  $2\ell_2 = 1 - \varepsilon$  in order to work with divergences. Again due to  $\Gamma(-1 + \varepsilon)$  the previous expression has simple zero at  $\varepsilon \rightarrow 0$ . The second operator in (5.2) commutes with  $z_2$  so that we have to calculate its action on the function which depends on variable  $z_1$  only. The action of this operator is not trivial

$$\begin{aligned} e^{-z_2 \partial_1} \frac{\Gamma(z_1 \partial_1 - 1 + \varepsilon)}{\Gamma(z_1 \partial_1 - \ell_1 - \frac{1-\varepsilon}{2} - u)} e^{z_2 \partial_1} \Phi(z_1) &= \sum_{k=0}^{\infty} \frac{\Phi^{(k)}(z_2)}{k!} \frac{\Gamma(k - 1 + \varepsilon)}{\Gamma(k - \ell_1 - \frac{1-\varepsilon}{2} - u)} \cdot z_{12}^k = \\ &= \Phi(z_2) \frac{\Gamma(-1 + \varepsilon)}{\Gamma(-\ell_1 - \frac{1}{2} - u)} + \Phi'(z_2) \frac{\Gamma(\varepsilon)}{\Gamma(-\ell_1 + \frac{1}{2} - u)} \cdot z_{12} + O(\varepsilon^0) \end{aligned}$$

but we really have to extract singular part  $\sim \varepsilon^{-1}$  only

$$[\Phi(z_2) (-\ell_1 - \frac{1}{2} - u) + \Phi'(z_2) \cdot z_{21}] \cdot \frac{\Gamma(-1 + \varepsilon)}{\Gamma(-\ell_1 + \frac{1}{2} - u)}.$$

Now we see that the simple pole cancels out the simple zero. The only difference with the case  $\ell_2 = 0$  is that we have to take into account two simple poles of  $\Gamma$ -function instead of one there. The action of  $\mathbb{R}$ -operator on components of  $\Psi(z_1, z_2)$  can be represented in the following form

$$\begin{aligned}\phi(z_1) &\rightarrow \left[ \phi(z_1) \left( -\ell_1 - \frac{1}{2} - u \right) + \phi'(z_1) \cdot z_{12} \right] \cdot \frac{\Gamma \left( -\ell_1 - \frac{1}{2} + u \right)}{\Gamma \left( -\ell_1 + \frac{1}{2} - u \right)} \\ z_2 \psi(z_1) &\rightarrow \left[ z_1^2 \psi'(z_1) - 2\ell_1 z_1 \psi(z_1) - z_2 \left( z_1 \psi'(z_1) + \left( -\ell_1 + \frac{1}{2} + u \right) \psi(z_1) \right) \right] \cdot \frac{\Gamma \left( -\ell_1 - \frac{1}{2} + u \right)}{\Gamma \left( -\ell_1 + \frac{1}{2} - u \right)}\end{aligned}$$

In the basis  $\mathbf{e}_1 = -z_2, \mathbf{e}_2 = 1$  we have for  $\ell_1 = \ell$  and  $\ell_2 \rightarrow \frac{1}{2}$

$$\begin{aligned}\mathbb{R} \left( u | \ell, \frac{1}{2} \right) \mathbf{e}_1 &= -\frac{\Gamma \left( -\ell - \frac{1}{2} + u \right)}{\Gamma \left( -\ell + \frac{1}{2} - u \right)} \cdot \left[ \mathbf{e}_1 \left( z_1 \partial_1 - \ell + \frac{1}{2} + u \right) + \mathbf{e}_2 \left( z_1^2 \partial_1 - 2\ell z_1 \right) \right] \\ \mathbb{R} \left( u | \ell, \frac{1}{2} \right) \mathbf{e}_2 &= -\frac{\Gamma \left( -\ell - \frac{1}{2} + u \right)}{\Gamma \left( -\ell + \frac{1}{2} - u \right)} \cdot \left[ \mathbf{e}_1 \left( -\partial_1 \right) + \mathbf{e}_2 \left( u + \frac{1}{2} + \ell - z_1 \partial_1 \right) \right]\end{aligned}$$

and in operator form we finally obtain<sup>3</sup>

$$\mathbb{R} \left( u | \ell, \frac{1}{2} \right) |_{\mathbb{V}_1} = -\frac{\Gamma \left( -\ell - \frac{1}{2} + u \right)}{\Gamma \left( -\ell + \frac{1}{2} - u \right)} \cdot \mathbf{L} \left( u + \frac{1}{2} | \ell \right). \quad (5.4)$$

Our calculations clearly demonstrate the cutting mechanism for the degree of polynomials, i.e. why at special points  $\ell = \frac{n}{2}$  there appear finite-dimensional invariant subspaces of the  $\mathbb{R}$ -operator.

Note that it is possible to use the reverse order of our two basic operators in the expression for  $\mathbb{R}$ -operator (2.12). Then everything is finite and there is no need in any regularization. The cutting mechanism is hidden in pure combinatorial manipulations using the Pfaff-Saalschütz formula so that we have chosen the presented method of calculations as the most illuminating.

And the last but not least: one of the reasons why we present here these calculations is to show that only for the whole  $\mathbb{R}$ -operator there exists the finite-dimensional invariant subspaces for special values of spins but operators  $\mathbf{R}^1$  or  $\mathbf{R}^2$  separately map beyond these subspaces. Keeping this in mind it will be not very surprising to encounter this phenomenon on the level of  $\mathbf{Q}$ -operators.

## 5.2 Restriction of general $\mathbb{R}$ -operator to finite-dimensional representations

In the previous subsection we have demonstrated by explicit calculation how to restrict the general  $\mathbb{R}$ -operator acting in the space  $\mathbb{U}_{-\ell_1} \otimes \mathbb{U}_{-\ell_2}$  to the invariant subspace  $\mathbb{U}_{-\ell_1} \otimes \mathbb{V}_n$ . For integer values of  $2\ell_1 = n$  the reducibility appears in the first tensor factor and the reduction to the  $(n+1)$ -dimensional irreducible representation space can be done. It is convenient to use the projection operators

$$\Pi_i^n z_i^k = z_i^k, \quad k \leq n \quad ; \quad \Pi_i^n z_i^k = 0, \quad k > n. \quad (5.5)$$

We shall write in boldface style the operators related to the integer or half-integer spin case which action is restricted to the finite-dimensional irreducible subspace

$$\mathbf{R}_{12} \left( u | \frac{n}{2}, \ell_2 \right) = \mathbb{R}_{12} \left( u | \frac{n}{2}, \ell_2 \right) \Pi_1^n \quad ; \quad \mathbf{R}_{12} \left( u | \ell_1, \frac{n}{2} \right) = \mathbb{R}_{12} \left( u | \ell_1, \frac{n}{2} \right) \Pi_2^n \quad (5.6)$$

The restriction in the second tensor factor is needed for construction of transfer matrices with finite-dimensional auxiliary space. In the previous section we have analyzed two particular cases needed for the derivation of the Baxter equation.

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<sup>3</sup>In this formula we use the notation  $\mathbf{L}(u|\ell)$  which contains explicitly spectral and spin parameter and which should not be mixed up with the other notation  $\mathbf{L}(u_1, u_2)$

### 5.2.1 Operator $\mathbf{R}$

Now we concentrate on the restriction in the first tensor factor which corresponds to the restriction in quantum space and consider the limit  $2\ell_1 \rightarrow n$ ,  $n = 0, 1, 2, \dots$  in the operator

$$\mathbf{R}(u|\ell_1, \ell_2) = e^{-z_1\partial_2} \frac{\Gamma(z_2\partial_2 - 2\ell_1)}{\Gamma(z_2\partial_2 - \ell_1 - \ell_2 - u)} e^{z_1\partial_2} \cdot e^{-z_2\partial_1} \frac{\Gamma(z_1\partial_1 - \ell_1 - \ell_2 + u)}{\Gamma(z_1\partial_1 - 2\ell_1)} e^{z_2\partial_1}. \quad (5.7)$$

We put  $2\ell_1 = n - \varepsilon$  and analyze the limit  $\varepsilon \rightarrow 0$  which is governed by the behaviour of the operator  $\Gamma(z\partial - n + \varepsilon)$  and its inverse for  $\varepsilon \rightarrow 0$ . The behaviour at small  $\varepsilon$  becomes more explicit after splitting operators into the contributions acting on the monomials  $z^k$  with  $k \leq n$  and  $k > n$  by means of the appropriate projectors (5.5):  $\Pi^n$  for monomials with  $k \leq n$  and  $1 - \Pi^n$  for monomials with  $k > n$  and using Euler's reflection formula  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$  for the contributions with projector  $\Pi^n$

$$\begin{aligned} \Gamma(z_2\partial_2 - n + \varepsilon) &= \Gamma(z_2\partial_2 - n + \varepsilon) (1 - \Pi_2^n) + \frac{\pi}{\sin \pi \varepsilon} \frac{(-1)^{z_2\partial_2 + n}}{\Gamma(1 + n - \varepsilon - z_2\partial_2)} \Pi_2^n, \\ \frac{1}{\Gamma(z_1\partial_1 - n + \varepsilon)} &= \frac{1}{\Gamma(z_1\partial_1 - n + \varepsilon)} (1 - \Pi_1^n) + \frac{\sin \pi \varepsilon}{\pi} (-1)^{z_1\partial_1 + n} \Gamma(1 + n - \varepsilon - z_1\partial_1) \Pi_1^n. \end{aligned}$$

The first contributions with projector  $1 - \Pi^n$  are nonsingular for  $\varepsilon \rightarrow 0$  and the second contribution clearly shows that the operator  $\Gamma(z_2\partial_2 - n + \varepsilon)$  diverges on monomials  $z_2^k, k \leq n$  and the operator  $\Gamma^{-1}(z_1\partial_1 - n + \varepsilon)$  annihilates all monomials  $z_1^k, k \leq n$  in the limit  $\varepsilon \rightarrow 0$ . We see that the nonsingular contribution with projector  $1 - \Pi^n$  does not contribute to a consistent restriction. The restriction to the subspace of monomials  $z_1^k, k \leq n$  leads to the operator  $\mathbf{R}_{12}(u|\frac{n}{2}, \ell_2)$  (5.6). Since  $e^{z_2\partial_1} \Pi_1^n = \Pi_1^n e^{z_2\partial_1} \Pi_1^n$  the projector  $\Pi_1^n$  extracts the corresponding contribution in the first factor in (5.7),

$$\begin{aligned} e^{-z_2\partial_1} \frac{\Gamma(z_1\partial_1 - \ell_1 - \ell_2 + u)}{\Gamma(z_1\partial_1 - 2\ell_1)} e^{z_2\partial_1} \Pi_1^n &= \\ &= \frac{(-1)^n \sin \pi \varepsilon}{\pi} \cdot e^{-z_2\partial_1} (-1)^{z_1\partial_1} \Gamma(z_1\partial_1 - \ell_1 - \ell_2 + u) \Gamma(1 + n - \varepsilon - z_1\partial_1) e^{z_2\partial_1} \Pi_1^n. \end{aligned}$$

It holds for arbitrary  $\varepsilon$ , but vanishes in the limit  $\varepsilon \rightarrow 0$ . Therefore one needs the singular contribution  $\sim \frac{1}{\varepsilon}$  from the second operator only

$$e^{-z_1\partial_2} \frac{\Gamma(z_2\partial_2 - 2\ell_1)}{\Gamma(z_2\partial_2 - \ell_1 - \ell_2 - u)} e^{z_1\partial_2} \rightarrow \frac{(-1)^n \pi}{\sin \pi \varepsilon} \cdot e^{-z_1\partial_2} \frac{(-1)^{z_2\partial_2} \Pi_2^n}{\Gamma(z_2\partial_2 - \ell_1 - \ell_2 - u) \Gamma(1 + n - \varepsilon - z_2\partial_2)} e^{z_1\partial_2}$$

Finally one obtains the following explicit expression for the restricted  $\mathbf{R}$ -operator

$$\begin{aligned} \mathbf{R}_{12}(u|\frac{n}{2}, \ell_2) &= \mathbf{P}_{12} \cdot e^{-z_1\partial_2} \frac{(-1)^{z_2\partial_2}}{\Gamma(z_2\partial_2 - \frac{n}{2} - \ell_2 - u) \Gamma(1 + n - z_2\partial_2)} \Pi_2^n e^{z_1\partial_2} \cdot \\ &\cdot e^{-z_2\partial_1} (-1)^{z_1\partial_1} \Gamma(z_1\partial_1 - \frac{n}{2} - \ell_2 + u) \Gamma(1 + n - z_1\partial_1) e^{z_2\partial_1} \Pi_1^n. \end{aligned} \quad (5.8)$$

Note that finite-dimensional subspace is preserved under the action of the operator  $\mathbf{R}_{12}(u|\frac{n}{2}, \ell_2)$  because projector  $\Pi_1^n$  appears not only on the right but on the left too

$$\mathbf{P}_{12} e^{-z_1\partial_2} \Pi_2^n = \mathbf{P}_{12} \Pi_2^n e^{-z_1\partial_2} \Pi_2^n = \Pi_1^n \mathbf{P}_{12} e^{-z_1\partial_2} \Pi_2^n. \quad (5.9)$$

### 5.2.2 Operators $\mathbf{R}^1$ , $\mathbf{R}^2$ and $\mathbf{S}$

Now we consider special reductions of (5.8) by specifying the values of the parameters  $v_1, v_2$ . We rewrite the expression for  $\mathbf{R}_{12}(u - v|\frac{n}{2}, \ell_2)$  using the parametrization

$$u_1 = u - \frac{n}{2} - 1, \quad u_2 = u + \frac{n}{2}; \quad v_1 = v - \ell_2 - 1, \quad v_2 = v + \ell_2, \quad (5.10)$$

and study the limits  $v_1 \rightarrow u_1$  or (and)  $v_2 \rightarrow u_2$ . We have  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2) = \mathbf{R}_{12}(u - v|\frac{n}{2}, \ell_2)$  so that

$$\begin{aligned} \mathbf{R}_{12}(u_1, u_2|v_1, v_2) &= P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1} \Gamma(z_1 \partial_1 + u_1 - v_2 + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n. \end{aligned} \quad (5.11)$$

- $v_2 \rightarrow u_2$

First we consider the limit of the operator  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2)$  when  $v_2 \rightarrow u_2$ . For this we put  $v_2 = u_2 - \delta$  and using Euler's reflection formula derive the leading contribution in the limit  $\delta \rightarrow 0$

$$(-1)^{z_1 \partial_1} \Gamma(z_1 \partial_1 + u_1 - u_2 + \delta + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) \rightarrow \frac{(-1)^{u_2 - u_1 - 1}}{\delta}$$

so that one obtains

$$\mathbf{R}_{12}(u_1, u_2|v_1, u_2 - \delta) \rightarrow \delta^{-1} \cdot \mathbf{R}_{12}^1(u_1|v_1, u_2),$$

where we defined some operator which is the relative of the operator  $\mathbb{R}_{12}^1$  (2.11) in the case of finite-dimensional representations

$$\mathbf{R}_{12}^1(u_1|v_1, u_2) \equiv P_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2 + u_2 - u_1 - 1}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2} \cdot \Pi_1^n \quad (5.12)$$

- $v_1 \rightarrow u_1$

Next we consider the limit of the operator  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2)$  for  $v_1 \rightarrow u_1$ . For this we put  $v_1 = u_1 + \delta$  and using Euler's reflection formula derive the leading contribution in the limit  $\delta \rightarrow 0$

$$\frac{(-1)^{z_2 \partial_2}}{\Gamma(z_2 \partial_2 + u_1 - \delta - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \rightarrow (-1)^{u_2 - u_1 - 1} \delta$$

so that one obtains

$$\mathbf{R}_{12}(u_1, u_2|u_1 + \delta, v_2) \rightarrow \delta \cdot \mathbf{R}_{12}^2(u_1, u_2|v_2),$$

where we defined the operator which is the relative of the operator  $\mathbb{R}_{12}^2$  (2.11) in the case of finite-dimensional representations

$$\begin{aligned} \mathbf{R}_{12}^2(u_1, u_2|v_2) &\equiv P_{12} \cdot e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \cdot \\ &\cdot e^{-z_2 \partial_1} (-1)^{z_1 \partial_1 + u_2 - u_1 - 1} \Gamma(z_1 \partial_1 + u_1 - v_2 + 1) \Gamma(u_2 - u_1 - z_1 \partial_1) e^{z_2 \partial_1} \Pi_1^n \end{aligned} \quad (5.13)$$

In comparison with (5.12) the last operator is more complicated.

- $v_1 \rightarrow u_1$  and  $v_2 \rightarrow u_2$

Finally we consider the limit of the operator  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2)$  when simultaneously  $v_1 \rightarrow u_1$  and  $v_2 \rightarrow u_2$ . We put  $v_1 = u_1 + \delta$  and  $v_2 = u_2 - \delta$  where  $\delta \rightarrow 0$ . As before we use Euler's reflection formula and obtain

$$\mathbf{R}_{12}(u_1, u_2|u_1 + \delta, u_2 - \delta) \rightarrow \mathbf{S}_{12} \equiv P_{12} \cdot e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \cdot \Pi_1^n \quad (5.14)$$

Contrary to the infinite-dimensional representation case,  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2)$  has the limit  $\mathbf{S}_{12}$  that is not equal to transposition. This result clearly demonstrates that in order to construct operators for finite-dimensional representation in the quantum space one has to perform carefully the appropriate limiting procedures.



### 5.2.3 Connection between two sets of operators

Now we are going to establish the connection between the operators  $\mathbf{R}_{12}^1(u_1|v_1, u_2)$  and  $\mathbf{R}_{12}^2(u_1, u_2|v_2)$  introduced in (5.12), (5.13) at special relations on parameters (5.10) and appropriate limits of operators  $\mathbb{R}_{12}^1(u_1|v_1, u_2)$ ,  $\mathbb{R}_{12}^2(u_1 u_2|v_2)$  (2.11).

We shift by  $\varepsilon$  the spin in quantum space:  $\ell = \frac{n}{2} \rightarrow \frac{n}{2} - \frac{\varepsilon}{2}$  and correspondingly  $u_1 = u - \frac{n}{2} - 1 \rightarrow u_1 + \frac{\varepsilon}{2}$  and  $u_2 = u + \frac{n}{2} \rightarrow u_2 - \frac{\varepsilon}{2}$  and consider the limit  $\varepsilon \rightarrow 0$ .

We start with  $\mathbb{R}_{12}^1(u_1 + \frac{\varepsilon}{2}|v_1, u_2 - \frac{\varepsilon}{2})$ . The leading contribution in the limit  $\varepsilon \rightarrow 0$  has the form

$$\mathbb{R}_{12}^1(u_1 + \frac{\varepsilon}{2}|v_1, u_2 - \frac{\varepsilon}{2}) \rightarrow \frac{1}{\varepsilon} \cdot \mathbf{P}_{12} \cdot e^{-z_1 \partial_2} \frac{(-1)^{z_2 \partial_2 + u_2 - u_1 - 1}}{\Gamma(z_2 \partial_2 + v_1 - u_2 + 1) \Gamma(u_2 - u_1 - z_2 \partial_2)} \Pi_2^n e^{z_1 \partial_2}.$$

The proof of this formula goes parallel to the above derivation of the expression for the operator  $\mathbf{R}_{12}(u|\frac{n}{2}, \ell_2)$ : we split  $\mathbb{R}_{12}^1(u_1 + \frac{\varepsilon}{2}|v_1, u_2 - \frac{\varepsilon}{2})$  using projectors  $\Pi_2^n$  and  $1 - \Pi_2^n$ , multiply by  $\varepsilon$  and obtain that in the limit  $\varepsilon \rightarrow 0$  only the singular contribution with  $\Pi_2^n$  survives.

The comparison with (5.12) gives another way for the calculation of  $\mathbf{R}_{12}^1(u_1|v_1, u_2)$  starting directly from his ancestor  $\mathbb{R}_{12}^1(u_1|v_1, u_2)$

$$\mathbf{R}_{12}^1(u_1|v_1, u_2) = \lim_{\varepsilon \rightarrow 0} \varepsilon \cdot \mathbb{R}_{12}^1(u_1 + \frac{\varepsilon}{2}|v_1, u_2 - \frac{\varepsilon}{2}) \Pi_1^n \quad (5.15)$$

The last formula means that  $\mathbb{R}_{12}^1(u_1|v_1, u_2)$  after renormalization in the limit  $\ell_1 \rightarrow \frac{n}{2}$  does not map beyond the subspace  $\mathbb{V}_n \otimes \mathbb{U}_{-\ell_2}$ . Moreover from (5.9) one can see that it maps  $\mathbb{U}_{-\frac{n}{2}} \otimes \mathbb{U}_{-\ell_2}$  to  $\mathbb{V}_n \otimes \mathbb{U}_{-\ell_2}$ .

Now we turn to the next pair of operators. This time the relation is more complicated:  $\mathbf{R}_{12}^2(u_1, u_2|v_2)$  can be obtained from the operator  $\mathbb{R}_{12}^2(u_1, u_2|v_2)$  but there exists the nontrivial first factor

$$\mathbb{S}_{12} \equiv \mathbf{P}_{12} e^{-z_1 \partial_2} \Pi_2^n e^{z_1 \partial_2} \quad (5.16)$$

so that the final formulae of connection is

$$\mathbf{R}_{12}^2(u_1, u_2|v_2) = \mathbb{S}_{12} \cdot \mathbf{P}_{12} \cdot \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \cdot \mathbb{R}_{12}^2(u_1 + \frac{\varepsilon}{2}, u_2 - \frac{\varepsilon}{2}|v_2) \Pi_1^n \quad (5.17)$$

The proof is very similar to the proof of (5.15). This formula means that the renormalized operator  $\mathbb{R}_{12}^2(u_1, u_2|v_2)$  in the limit  $\ell_1 \rightarrow \frac{n}{2}$  maps beyond the subspace  $\mathbb{V}_n \otimes \mathbb{U}_{-\ell_2}$  and therefore the correcting operator  $\mathbb{S}_{12}$  is indispensable. Indeed  $\mathbb{S}_{12}$  maps  $\mathbb{U}_{-\frac{n}{2}} \otimes \mathbb{U}_{-\ell_2}$  to  $\mathbb{V}_n \otimes \mathbb{U}_{-\ell_2}$  as one can see from (5.9).

Let us point out the connection between the double reduction of  $\mathbf{R}_{12}(u_1, u_2|v_1, v_2)$  (5.14) and operator  $\mathbb{S}_{12}$  (5.16)

$$\mathbf{S}_{12} = \mathbb{S}_{12} \cdot \Pi_1^n$$

## 5.3 The general transfer matrices and Q-operators

After the necessary preparations in the previous section we proceed to the construction of the general transfer matrix (4.1) from the new building blocks  $\mathbf{R}_{k0}$ . Here we meet a certain difficulty: the trace over the infinite-dimensional auxiliary space  $\mathbb{C}[z_0]$  diverges. Consequently we have to introduce some kind of regularization. We shall use the following regularization

$$\text{tr}_0 \mathbf{B} \longrightarrow \text{tr}_0 q^{z_0 \partial_0} \mathbf{B} \quad , \quad |q| < 1$$

which corresponds to quasiperiodic boundary conditions of the spin chain. A shortcoming of this regularization is a violation of  $s\ell_2$ -symmetry in the traces and correspondingly in the transfer matrices. Thus we define the general transfer matrix for integer or half-integer values of  $\ell$  and  $|q| < 1$  as

$$\mathbf{T}_s(u) = \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}(u|\frac{n}{2}, s) \mathbf{R}_{20}(u|\frac{n}{2}, s) \cdots \mathbf{R}_{N0}(u|\frac{n}{2}, s) . \quad (5.18)$$

This operator is well defined on the finite-dimensional quantum space of the chain.

Now we consider the factorization of the general transfer matrix (5.18). We start as before with the three term relation (4.7) and restrict it at site  $k$  to  $\mathbb{V}^n \otimes \mathbb{C}[z_0] \otimes \mathbb{C}[z_{0'}]$  for  $\ell = \frac{n}{2}$

$$\begin{aligned} & \mathbb{R}_{00'}^2(v_1, v_2|w_2) \mathbf{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbf{R}_{k0}(u_1, u_2|w_1, v_2) \mathbf{R}_{k0'}(u_1, u_2|v_1, w_2) \mathbb{R}_{00'}^2(v_1, v_2|w_2). \end{aligned} \quad (5.19)$$

Then we specify parameters as  $w_1 = u_1 + \delta$ ,  $w_2 = u_2 - \delta$  and obtain in the limit  $\delta \rightarrow 0$  keeping in mind the results of Section 5.2.2

$$\mathbb{R}_{00'}^2(v_1, v_2|v_2) \mathbf{S}_{k0'} \mathbf{R}_{k0}(u_1, u_2|v_1, v_2) = \mathbf{R}_{k0}^2(u_1, u_2|v_2) \mathbf{R}_{k0'}^1(u_1|v_1, u_2) \mathbb{R}_{00'}^2(v_1, v_2|u_2).$$

This local relation leads in the standard way to the factorization relation for the corresponding regularized transfer matrices if we take into account that  $[q^{z_0 \partial_0 + z_{0'} \partial_{0'}}, \mathbb{R}_{00'}] = 0$

$$\begin{aligned} & \text{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} \mathbf{S}_{10'} \mathbf{S}_{20'} \cdots \mathbf{S}_{N0'} \right] \cdot \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbf{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbf{R}_{N0}(u_1, u_2|v_1, v_2) \right] = \\ & = \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbf{R}_{10}^2(u_1, u_2|v_2) \cdots \mathbf{R}_{N0}^2(u_1, u_2|v_2) \right] \cdot \text{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} \mathbf{R}_{10'}^1(u_1|v_1, u_2) \cdots \mathbf{R}_{N0'}^1(u_1|v_1, u_2) \right] \end{aligned}$$

The second case of factorization can be obtained in a similar way. After introduction of the notation for the transfer matrices

$$\begin{aligned} \mathbf{Q}_1(u - v_1) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^1(u_1|v_1, u_2) \cdots \mathbf{R}_{N0}^1(u_1|v_1, u_2), \\ \mathbf{Q}_2(u - v_2) &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{R}_{10}^2(u_1, u_2|v_2) \cdots \mathbf{R}_{N0}^2(u_1, u_2|v_2), \\ \mathbf{S} &= \text{tr}_0 q^{z_0 \partial_0} \mathbf{S}_{10} \mathbf{S}_{20} \cdots \mathbf{S}_{N0}, \end{aligned} \quad (5.20)$$

we can rewrite factorization relations in a compact form

$$\mathbf{S} \mathbf{T}_s(u) = \mathbf{Q}_2(u - s) \mathbf{Q}_1(u + s + 1) = \mathbf{Q}_1(u + s + 1) \mathbf{Q}_2(u - s). \quad (5.21)$$

This construction is analogous to the construction from the first part, where  $\ell$  was a generic complex number. The proof of the commutativity

$$[\mathbf{T}_s(u), \mathbf{Q}_k(v)] = 0 \quad ; \quad [\mathbf{Q}_i(u), \mathbf{Q}_k(v)] = 0 \quad ; \quad [\mathbf{S}, \mathbf{Q}_k(u)] = 0 \quad ; \quad [\mathbf{S}, \mathbf{T}_s(u)] = 0. \quad (5.22)$$

uses the general Yang-Baxter equation and also goes parallel the corresponding derivation given in the first part.

#### 5.4 Connection between Q-operators of compact and generic spin and the Baxter equations

In Section 5.2.3 we have established relations between the two sets of R-operators. Now we formulate the corresponding relations for the transfer matrices. Let us consider the relation between the operators  $\mathbf{Q}_1(u)$  and  $\mathbf{Q}_2(u)$  and the limits at  $\ell \rightarrow \frac{n}{2}$  of the regularized Baxter Q-operators  $\mathbf{Q}_1(u|q)$  and  $\mathbf{Q}_2(u|q)$  (C.5), which are trivial modifications of operators  $\mathbf{Q}_{1,2}(u)$  being constructed in sect.4.

The connection between operators  $\mathbf{Q}_1(u)$  and  $\mathbf{Q}_1(u|q)$  can be established directly using the formula (5.15): we put the spin  $\ell = \frac{n}{2} - \frac{\varepsilon}{2}$  in quantum space and in the limit  $\varepsilon \rightarrow 0$  obtain

$$\mathbf{Q}_1(u - v_1) = \lim_{\varepsilon \rightarrow 0} \varepsilon^N \cdot \mathbf{Q}_1(u - v_1|q) \Big|_{\ell = \frac{n-\varepsilon}{2}} \cdot \Pi^n \quad (5.23)$$

where  $\Pi^n \equiv \Pi_1^n \Pi_2^n \cdots \Pi_N^n$ .

In our notations the boldface style in  $\mathbf{Q}_1(u - v_1)$  means that the parameter of the spin in this operator is half-integer  $\ell = \frac{n}{2}$  and the operator is restricted to the appropriate finite-dimensional subspace by the projector  $\Pi^n$ . But it is easy to see that there exists the extension of operator  $\mathbf{Q}_1(u - v_1)$  to the whole space of polynomials or in other words there exists the limit

$$\mathbf{Q}_1(u - v_1) = \lim_{\varepsilon \rightarrow 0} \varepsilon^N \cdot \mathbf{Q}_1(u - v_1|q)|_{\ell=\frac{n-\varepsilon}{2}}. \quad (5.24)$$

To avoid making copies of notations we denote this extended operator by the same notations as in the case of generic spin. We hope it will not lead to misunderstanding and it will be clear from the context which operator is used. The operator  $\mathbf{Q}_1(u - v_1)$  inherits properties from its local building blocks. In particular, due to (5.9) this operator does not map beyond the finite-dimensional quantum space and moreover it maps the whole space of polynomials to the finite-dimensional quantum space.

To derive the second relation we rely on the factorization (A.2) for arbitrary spin in quantum space. Now we need a simple modification of this formula because of the regularization

$$\begin{aligned} & \text{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} P_{10'} \cdots P_{N0'} \right] \cdot \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2) \right] = \\ & = \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}^1(u_1|v_1, u_2) \cdots \mathbb{R}_{N0}^1(u_1|v_1, u_2) \right] \cdot \text{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} \mathbb{R}_{10'}^2(u_1, u_2|v_2) \cdots \mathbb{R}_{N0'}^2(u_1, u_2|v_2) \right]. \end{aligned} \quad (5.25)$$

Next we multiply from the right by the projector  $\Pi^n = \Pi_1^n \Pi_2^n \cdots \Pi_N^n$  which results in the restriction to the appropriate subspace of the quantum space and put  $2\ell = n - \varepsilon$  or equivalently  $u_1 - u_2 + 1 = -n + \varepsilon$ . It remains to do the limit  $\varepsilon \rightarrow 0$ . In the left hand side one obtains the operator

$$P q^{z_1 \partial_1} \cdot \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbf{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbf{R}_{N0}(u_1, u_2|v_1, v_2) \right]$$

where we used  $\text{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} P_{10'} \cdots P_{N0'} \right] = P q^{z_1 \partial_1}$ . In the right hand side we have to do some rearrangement of  $\varepsilon$  to be sure that a finite limit exists for both operators in the product

$$\mathbf{Q}_1(u - v_1|q) \cdot \mathbf{Q}_2(u - v_2|q) \Pi^n = \varepsilon^N \mathbf{Q}_1(u - v_1|q) \cdot \frac{1}{\varepsilon^N} \mathbf{Q}_2(u - v_2|q) \Pi^n$$

Note that in the limit  $\varepsilon \rightarrow 0$  the first operator  $\varepsilon^N \mathbf{Q}_1(u - v_1|q)$  gives operator  $\mathbf{Q}_1(u - v_1)$  (5.24), the extended version of the operator  $\mathbf{Q}_1(u - v_1)$ . It is easy to see that the expression  $\varepsilon^{-N} \cdot \mathbf{Q}_2(u - v_2|q) \cdot \Pi^n$  also gives a well defined operator in the limit  $\varepsilon \rightarrow 0$ . We shall study this operator in detail in next subsection. Note, that contrary to the previous case of operator  $\mathbf{Q}_1(u)$  the extension of the operator  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} \cdot \mathbf{Q}_2(u - v_2|q) \Pi^n$  to whole space does not exist. The reason is that for finite  $\varepsilon$  the result of the action of  $\mathbf{Q}_2$  on vectors of the subspace extracted by the projector  $\Pi^n$  is  $\sim \varepsilon^N$  but  $\sim 1$  or lower powers of  $\varepsilon$  for action on the complementary subspace extracted by projector  $1 - \Pi^n$ . Due to this fact after multiplication by  $\varepsilon^{-N}$  one obtains finite results in the first case and divergences in the second case.

We obtain the following relation

$$P q^{z_1 \partial_1} \cdot \text{tr}_0 \left[ q^{z_0 \partial_0} \mathbf{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbf{R}_{N0}(u_1, u_2|v_1, v_2) \right] = \mathbf{Q}_1(u - v_1) \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} \mathbf{Q}_2(u - v_2|q) \Pi^n,$$

Then we have to specify  $v_1 = u_1 + \delta$  and take the limit  $\delta \rightarrow 0$ . Finally we obtain

$$P q^{z_1 \partial_1} \cdot \mathbf{Q}_2(u - v_2) = S \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} \mathbf{Q}_2(u - v_2|q) \cdot \Pi^n, \quad (5.26)$$

where the new operator  $S$  is the special limit of the operator  $\mathbf{Q}_1$  (5.24) and can be represented as a transfer matrix constructed from the operators  $\mathbb{S}_{k0}$  (5.16)

$$S = \lim_{\delta \rightarrow 0} \delta^{-N} \mathbf{Q}_1\left(\frac{n}{2} + 1 - \delta\right) = \text{tr}_0 q^{z_0 \partial_0} \mathbb{S}_{10} \mathbb{S}_{20} \cdots \mathbb{S}_{N0}. \quad (5.27)$$

Like  $Q_1$  this operator  $S$  maps the whole space of polynomials to the finite-dimensional subspace. We see that for the operator  $Q_2(u)$  the connection is more complicated: it coincides up to normalization with the product of the nontrivial operator  $q^{-z_1\partial_1} P^{-1} S$  and the restriction of  $Q_2(u|q)$  to the invariant subspace appearing for  $\ell = \frac{n}{2}$ . (5.26) is a global analogue of the local formula (5.17). Above we have seen on particular example (5.1) that the renormalized Baxter operator  $Q_2(u)$  maps beyond the finite-dimensional subspace. The same is true for  $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-N} Q_2(u|q) \cdot \Pi^n$ . Now we see that the problem of finite-dimensional representations is resolved by means of the special operator  $q^{-z_1\partial_1} P^{-1} S$ . Its role is to reverse the mapping back to the finite-dimensional subspace after it went beyond by the action of the limit of  $Q_2(u|q)$  producing the correct Baxter operator  $Q_2(u)$  on the finite-dimensional quantum space.

The relations (5.23) and (5.26) allow to obtain explicit compact formulae for Baxter operators  $Q_{1,2}(u)$ . We postpone this derivation to the Section 5.5.

Now we turn to Baxter relations for  $Q_1(u)$  and  $Q_2(u)$ . Both proofs of Baxter relation presented above in sections 3 and 4 can be transferred step by step to the case of half-integer  $\ell$ . However there is a simpler way to establish such relations since we know the relations (5.23) and (5.26) between Baxter operators for finite and infinite-dimensional quantum spaces. We follow this way below.

The introduction of the regularization leads to simple modifications which are discussed in detail in Appendix C. Finally the Baxter equations for regularized operators  $Q_{1,2}(u|q)$  have the form

$$t(u|q) Q_1(u|q) = Q_1(u+1|q) + q \cdot (u_1 u_2)^N \cdot Q_1(u-1|q), \quad (5.28)$$

$$t(u|q) Q_2(u|q) = q \cdot Q_2(u+1|q) + (u_1 u_2)^N \cdot Q_2(u-1|q), \quad (5.29)$$

where

$$t(u|q) = \text{tr} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} L_1(u) L_2(u) \cdots L_N(u). \quad (5.30)$$

In order to derive Baxter equation for  $Q_1$  we have to multiply the obtained equation (5.28) by the projector  $\Pi^n$  from the right and specify the spin parameter  $\ell = \frac{n}{2}$

$$t(u|q) Q_1(u) = Q_1(u+1) + q \cdot (u_1 u_2)^N \cdot Q_1(u-1). \quad (5.31)$$

The Baxter equation for operator  $Q_2$ :

$$t(u|q) Q_2(u) = q \cdot Q_2(u+1) + (u_1 u_2)^N \cdot Q_2(u-1) \quad (5.32)$$

is derived in a similar way from (5.29) but there is one additional step - the multiplication by the operator  $q^{-z_1\partial_1} P^{-1} S$  from the left. This operator does not depend on spectral parameter so that this step does not change the equation.

Thus we see that  $Q_1$  and  $Q_2$  possess all expected properties of Baxter operators. Moreover because they are obtained at special values of parameters from the general transfer matrix (5.18) they map finite-dimensional space into itself. However they are not  $sl_2$ -invariant because of  $q$ -regularization.

## 5.5 Explicit action on polynomials

Now we are going to consider explicit formulae for the action of the constructed operators  $Q_k(u)$  on polynomials. For this purpose we shall use the connection with the operators  $Q_k(u|q)$  on infinite quantum space.

It is easy to repeat step by step all derivations from the section 4.3 and obtain formulae for the operators  $Q_k(u|q)$  which simply mimic formulae for the operators  $Q_k(u)$  with needed minimal

modifications due to q-regularization. The expression for the action of the operator  $Q_2(u|q)$  on polynomials is the following

$$Q_2(u|q) \Psi(\vec{z}) = P \cdot R_2(\lambda_1 \partial_{\lambda_1}) \cdots R_2(\lambda_N \partial_{\lambda_N})|_{\lambda=1} \cdot \Psi(\Lambda_q \vec{z}), \quad (5.33)$$

where

$$\Lambda_q = \begin{pmatrix} q\lambda_1 & 1-\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 1-\lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & 1-\lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_{N-1} & 1-\lambda_{N-1} & 0 \\ 1-\lambda_N & 0 & 0 & 0 & \dots & \lambda_N \end{pmatrix}; \quad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \dots \\ z_N \end{pmatrix}$$

and the expression for the action of the operator  $Q_1(u|q)$  on polynomials is very similar to the corresponding formula for  $Q_2(u|q)$

$$Q_1(u|q) \Psi(\vec{z}) = R_1(\lambda_1 \partial_{\lambda_1}) \cdots R_1(\lambda_N \partial_{\lambda_N})|_{\lambda=1} \cdot \frac{1}{1 - q\bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}), \quad (5.34)$$

where  $\bar{\lambda} \equiv 1 - \lambda$  and

$$\Lambda_q' = \begin{pmatrix} 1 - \frac{1}{\lambda_1} & \frac{1}{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & 1 - \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & 1 - \frac{1}{\lambda_3} & \frac{1}{\lambda_3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 - \frac{1}{\lambda_{N-1}} & \frac{1}{\lambda_{N-1}} & 0 \\ \frac{1}{q\lambda_N} & 0 & 0 & 0 & \dots & 1 - \frac{1}{\lambda_N} \end{pmatrix}.$$

Formulae (5.33) and (5.34) are the starting points for the derivation of various representations for the Q-operators. Integral formulae similar to (4.22) and (4.23) are obtained by evident changes in matrices  $\Lambda$  so that we shall not repeat all these formulae but instead concentrate on the derivation of the useful representation for operator  $Q_1$  in the case of half-integer spin.

We use the following integral representation for all operators  $R_1(\lambda_k \partial_{\lambda_k})$

$$R_1(\lambda \partial_{\lambda}) \Phi(\lambda)|_{\lambda=1} = \frac{1}{\Gamma(1 + \ell - u)} \cdot \int_0^1 d\lambda (1 - \lambda)^{\ell-u} \lambda^{-2\ell-1} \Phi(\lambda).$$

Note that for  $2\ell = n - \varepsilon$  we have the pole  $\sim \frac{1}{\varepsilon}$  due to divergence in the integral arising from the pole  $\lambda^{-n-1}$  in the integrand at  $\varepsilon = 0$ . Because of the factor  $\varepsilon^N$  in the definition of the operator  $Q_1(u)$  (5.24) we have to calculate only the singular contribution  $\sim \frac{1}{\varepsilon}$  in each integral resulting in a significant simplification of the calculation.

$$\int_0^1 d\lambda (1 - \lambda)^{\ell-u} \lambda^{-2\ell-1} \Phi(\lambda) \rightarrow \frac{1}{\varepsilon} \frac{\partial_{\lambda}^n}{n!} (1 - \lambda)^{\frac{n}{2}-u} \Phi(\lambda) \Big|_{\lambda=0}$$

Finally we arrive at a compact formula for the action of the operator  $Q_1(u)$

$$Q_1(u) \Psi(\vec{z}) = \frac{1}{\Gamma^N(1 + \frac{n}{2} - u) n!^N} \cdot \partial_{\lambda_1}^n \cdots \partial_{\lambda_N}^n \frac{(\bar{\lambda}_1 \cdots \bar{\lambda}_N)^{\frac{n}{2}-u}}{1 - q\bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}) \Big|_{\lambda=0}. \quad (5.35)$$

This operator is defined on the whole infinite-dimensional quantum space. In order to calculate the action of  $Q_1(u)$  on arbitrary polynomial  $\Psi(\vec{z})$  according to (5.23) one has to pick up powers of  $z_1, \dots, z_N$  less or equal  $n$  and then apply formula (5.35) to the obtained polynomial.

At the point of degeneracy  $u = \frac{n}{2} + 1 - \delta$  in the appropriate limit  $\delta \rightarrow 0$  the operator  $\mathbf{Q}_1(u)$  reduces to the operator  $\mathbf{S}$  (5.27)

$$\mathbf{S}\Psi(\vec{z}) = \left(\frac{1}{n!}\right)^N \partial_{\lambda_1}^n \cdots \partial_{\lambda_N}^n \frac{1}{\bar{\lambda}_1 \cdots \bar{\lambda}_N} \frac{1}{1 - q\bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}) \Big|_{\lambda=0} \quad (5.36)$$

This operator as well as  $\mathbf{Q}_1(u)$  is defined on the whole infinite-dimensional quantum space. It is evident how to calculate  $\mathbf{S}$  (5.20) on arbitrary polynomial  $\Psi(\vec{z})$  because  $\mathbf{S} = \mathbf{S} \cdot \Pi^n$ .

The expression (5.36) can be derived directly. At first let us establish how  $\mathbb{S}_{k0}$  (5.16) acts on the function of two arguments

$$\begin{aligned} \mathbb{S}_{k0} \Phi(z_k, z_0) &= \mathbf{P}_{k0} e^{-z_k \partial_0} \Pi_0^n \Phi(z_k, z_0 + z_k) = \mathbf{P}_{k0} \sum_{m=0}^n \frac{z_{0k}^m}{m!} \partial_z^m \Phi(z_k, z) \Big|_{z=z_k} = \\ &= \sum_{m=0}^n \frac{\partial_{\lambda}^m}{m!} \Phi(z_0, z_0 + \lambda z_{k0}) \Big|_{\lambda=0} = \sum_{m=0}^n \frac{\partial_{\lambda}^m}{m!} \lambda^{z_{k0} \partial_0} \Phi(z_0, z_k) \Big|_{\lambda=0}. \end{aligned}$$

Thus we have

$$\mathbb{S}_{k0} = \mathbf{e}_n(\partial_{\lambda}) \cdot \mathbf{P}_{k0} \cdot \lambda^{z_{0k} \partial_0} \Big|_{\lambda=0} \quad ; \quad \mathbf{e}_n(\partial_{\lambda}) \equiv \sum_{m=0}^n \frac{\partial_{\lambda}^m}{m!},$$

and consequently  $\mathbf{S}$  (5.27) takes the form

$$\mathbf{S} = \mathbf{e}_n(\partial_{\lambda_1}) \cdots \mathbf{e}_n(\partial_{\lambda_N}) \text{tr}_{\mathbb{V}_0} \left[ q^{z_0 \partial_0} \mathbf{P}_{10} \lambda_1^{z_{01} \partial_0} \cdots \mathbf{P}_{N0} \lambda_N^{z_{0N} \partial_0} \right] \Big|_{\lambda=0}.$$

The involved trace is calculated by using (4.17)

$$\mathbf{S} \Psi(\vec{z}) = \mathbf{e}_n(\partial_{\lambda_1}) \cdots \mathbf{e}_n(\partial_{\lambda_N}) \frac{1}{1 - q\bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}) \Big|_{\lambda=0} \quad (5.37)$$

It remains to note that

$$\mathbf{e}_n(\partial_{\lambda}) \Psi(\lambda) \Big|_{\lambda=0} = \frac{\partial_{\lambda}^n}{n!} \frac{1}{1 - \lambda} \Psi(\lambda) \Big|_{\lambda=0}$$

so that we are coming back to formula (5.36).

Finally we change normalization of Baxter operators  $\mathbf{Q}_1(u)$  and  $\mathbf{Q}_2(u)$  in order to make them to become polynomials in the spectral parameter  $u$ . The explicit action of the renormalized  $\mathbf{Q}_1(u)$  (5.35) which we denote  $\mathbf{P}(u)$  has the form

$$\mathbf{P}(u) \Psi(\vec{z}) = \partial_{\lambda_1}^n \cdots \partial_{\lambda_N}^n \frac{(\bar{\lambda}_1 \cdots \bar{\lambda}_N)^{\frac{n}{2}-u}}{1 - q\bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\Lambda_q'^{-1} \vec{z}) \Big|_{\lambda=0} \quad (5.38)$$

where  $\Psi(\vec{z})$  is polynomial from finite-dimensional invariant subspace.

The explicit action of the renormalized  $\mathbf{Q}_2(u)$  which we denote  $\mathbf{Q}(u)$  on the generating function of finite-dimensional representation is

$$\begin{aligned} \mathbf{Q}(u) : (1 - x_1 z_1)^n \cdots (1 - x_N z_N)^n &\mapsto \\ \mapsto \mathbf{S} \cdot (1 - x_1 z_1)^{\frac{n}{2}-u} (1 - x_1 z_2)^{\frac{n}{2}+u} \cdots &(1 - x_N z_N)^{\frac{n}{2}-u} (1 - x_N q^{-1} z_1)^{\frac{n}{2}+u} \end{aligned} \quad (5.39)$$

and follows from (5.26) and the explicit action for the renormalized  $\mathbf{Q}_2(u|q)$  (C.13).

## 6 Discussion

We have analyzed the set of commuting operators of the closed homogeneous spin chain, where the quantum states on the sites are representations of  $sl_2$  either infinite-dimensional for generic spin values or finite-dimensional for integer or half-integer spins. These operators and their spectra contain all information on the quantum system. The aim of studying the relations between them, the Baxter relations in particular, is to obtain this information in a most convenient and explicit form.

Comparing the ordinary transfer matrix  $t(u)$ , its generalizations  $t_n(u)$ ,  $T_s(u)$  and the Baxter operators  $Q(u)$  a simple systematics in their construction is observed resulting in an understanding of their relations. All they are constructed as traces of products of operators with one factor for each chain site. The factor operators act on the tensor product of the quantum and the auxiliary spaces. Performing construction for infinite-dimensional representations in the quantum space at generic spin  $\ell$  we have seen that in the most general case the factor at site  $k$  is the general Yang-Baxter operator  $\mathbb{R}_{k0}$ . In the other cases the factor operators are certain reductions obtained therefrom by imposing conditions on the representation parameters  $v_1, v_2$  referring to the auxiliary space:

<i>chain operator</i>	<i>site operator</i>	<i>restriction</i>
$T_s$	$\mathbb{R}_{k0}$	—
$t$	$L_k \sim \mathbf{R}_{k0}(\ell, \frac{1}{2})$	$v_2 - v_1 = 2$ and $\Pi_0^1$
$t_n$	$\mathbf{R}_{k0}(\ell, \frac{n}{2})$	$v_2 - v_1 = n + 1$ and $\Pi_0^n$
$Q_1$	$\mathbb{R}_{k0}^1$	$v_2 = u_2$
$Q_2$	$\mathbb{R}_{k0}^2$	$v_1 = u_1$
$P$	$P_{k0}$	$v_1 = u_1$ and $v_2 = u_2$

Our proofs of factorization and commutativity for different transfer matrices rely on local three-term relations – Yang-Baxter relations. We consider the derivation of these factorizations as one of the main results due to its transparency and simplicity. An appropriate case of such relations describes the intertwining of chain site operators. The standard argument going parallel to the proof of the ordinary transfer matrix commutativity then leads to the desired relation for the chain operators. In particular we have demonstrated how to deduce algebraic properties of Baxter operators only from these relations without any references to other concepts.

We have presented two ways of deriving the Baxter relations. The systematics applies in analogy also to the case of integer or half-integer spin  $\ell = \frac{n}{2}$  with finite-dimensional representation spaces at the sites. Here the site operators are  $\mathbf{R}_{k0}(u|\frac{n}{2}, s)$  in the case of the general transfer matrix  $T_s$  the Yang-Baxter operators restricted to the irreducible subspace by means of projector  $\Pi_k^n$  at  $u_2 - u_1 = n + 1$ . Additional restrictions on parameter  $v_1, v_2$  lead to the other reductions:

<i>chain operator</i>	<i>site operator</i>	<i>additional restriction</i>
$T_s$	$\mathbf{R}_{k0}(\frac{n}{2} s)$	—
$Q_1$ or $P$	$\mathbf{R}_{k0}^1$	$v_2 = u_2$
$Q_2$ or $Q$	$\mathbf{R}_{k0}^2$	$v_1 = u_1$
$S$	$\mathbb{S}_{k0}\Pi_k^n$	$v_1 = u_1$ and $v_2 = u_2$

The presented proofs of factorization and commutativity of transfer matrices for finite-dimensional representations are completely parallel to the corresponding proofs of the infinite-dimensional case. Our analysis clarifies the relation between the cases of generic spins and half-integer or integer spins. We have seen that the naive attempt to substitute integer or half-integer values for the spin parameter  $\ell = \frac{n}{2}$  in the general expressions for the regularized transfer matrices meets difficulties in cases related to the operator  $Q_2$  just because this operator maps beyond the finite-dimensional

quantum space, whereas the general transfer matrix  $T_s$  and the operator  $Q_1$  do not map beyond the finite-dimensional quantum space. Careful calculations of limits  $\ell = \frac{n}{2}$  in the general formulae produces the transfer matrix  $S$  constructed from local operators  $S_{k0}$  being the nontrivial analogue of the permutation  $P_{k0}$  appearing in the finite-dimensional case. The operator  $S$  is a reduction of Baxter operator  $Q_1$  and it improves the action of the operator  $Q_2$ . Due to the special operator  $S$  appearing naturally in our scheme we obtain the desired set of Baxter operators acting in finite-dimensional quantum space.

Moreover, besides presenting general formulae for Baxter operators and proving their algebraic properties we present explicit compact expressions for their action on polynomials. Such expressions in the case of finite-dimensional quantum space are particularly simple and appropriate for practical calculations.

Despite of similarities in our considerations of infinite-dimensional and finite-dimensional cases there are significant differences between them. In the infinite-dimensional case the Baxter operators construction is  $sl_2$  symmetric whereas in finite-dimensional case we had to introduce a symmetry breaking regularization of divergent traces by means of  $q$ -regularization. We consider this as considerable shortcoming of the presented construction in the finite-dimensional case. Actually we have reasons to expect that it should be possible to construct Baxter operators also in the case of integer or half-integer spin preserving the symmetry of the model.

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## Appendices

### A Factorization and commutativity

The general transfer matrix constructed from operators  $\mathbb{R}_{k0}(u_1, u_2|v_1, v_2)$  factorizes into the product of two transfer matrices constructed from operators  $\mathbb{R}_{k0}^2(u_1, u_2|v_2)$  and  $\mathbb{R}_{k0'}^1(u_1|v_1, u_2)$ . This factorization for global objects follows in a clear and direct way from the local relations for their building blocks.

Let us accomplish analogous steps in order to obtain the second factorization. We rewrite (2.16) as follows

$$\begin{aligned} \mathbb{R}_{00'}^1(v_1|w_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ = \mathbb{R}_{k0}(u_1, u_2|v_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, v_2) \mathbb{R}_{00'}^1(v_1|w_1, w_2) \end{aligned} \quad (A.1)$$

and then specifying parameters as  $w_1 = u_1$  and  $w_2 = u_2$  we arrive at the intertwining relation

$$\mathbb{R}_{00'}^1(v_1|u_1, u_2) \cdot P_{k0'} \cdot \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \mathbb{R}_{k0}^1(u_1|v_1, u_2) \cdot \mathbb{R}_{k0'}^2(u_1, u_2|v_2) \mathbb{R}_{00'}^1(v_1|u_1, u_2)$$

which leads to the relation for the transfer matrices

$$\text{tr}_{0'} [P_{10'} \cdots P_{N0'}] \cdot \text{tr}_0 [\mathbb{R}_{10}(u_1, u_2|v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2|v_1, v_2)] = \quad (A.2)$$



$$= \text{tr}_0 \left[ \mathbb{R}_{10}^1(u_1|v_1, u_2) \cdots \mathbb{R}_{N0}^1(u_1|v_1, u_2) \right] \cdot \text{tr}_{0'} \left[ \mathbb{R}_{10'}^2(u_1, u_2|v_2) \cdots \mathbb{R}_{N0'}^2(u_1, u_2|v_2) \right].$$

The direct consequence of the two factorizations (4.8) and (A.2) is the commutativity of the transfer matrices constructed from  $\mathbb{R}^1$  and  $\mathbb{R}^2$ . However it is more instructive to derive commutativity from local intertwining relations. Below for completeness we list the necessary relations.

The Yang-Baxter equation (2.14) in the form

$$\begin{aligned} & \mathbb{R}_{00'}(v_1, v_2|w_1, w_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) = \\ & = \mathbb{R}_{k0}(u_1, u_2|v_1, v_2) \mathbb{R}_{k0'}(u_1, u_2|w_1, w_2) \mathbb{R}_{00'}(v_1, v_2|w_1, w_2) \end{aligned} \quad (\text{A.3})$$

leads to the commutativity of the general transfer matrices constructed from  $\mathbb{R}$ -operators. Then specifying parameters in (A) we get the following three relations. From the first one ( $v_1 = u_1$ ,  $w_2 = u_2$ )

$$\begin{aligned} & \mathbb{R}_{00'}(u_1, v_2|w_1, u_2) \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{00'}(u_1, v_2|w_1, u_2) \end{aligned} \quad (\text{A.4})$$

we obtain immediately the commutativity of the transfer matrices constructed from  $\mathbb{R}^1$  and  $\mathbb{R}^2$ . The second relation ( $u_2 = v_2 = w_2$ )

$$\begin{aligned} & \mathbb{R}_{00'}^1(v_1|w_1, u_2) \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{k0}^1(u_1|v_1, u_2) = \\ & = \mathbb{R}_{k0}^1(u_1|v_1, u_2) \mathbb{R}_{k0'}^1(u_1|w_1, u_2) \mathbb{R}_{00'}^1(v_1|w_1, u_2) \end{aligned} \quad (\text{A.5})$$

leads to commutativity of the transfer matrices constructed from  $\mathbb{R}^1$ , and the third one ( $u_1 = v_1 = w_1$ )

$$\begin{aligned} & \mathbb{R}_{00'}^2(u_1, v_2|w_2) \mathbb{R}_{k0'}^2(u_1, u_2|w_2) \mathbb{R}_{k0}^2(u_1, u_2|v_2) = \\ & = \mathbb{R}_{k0}^2(u_1, u_2|v_2) \mathbb{R}_{k0'}^2(u_1, u_2|w_2) \mathbb{R}_{00'}^2(u_1, v_2|w_2) \end{aligned} \quad (\text{A.6})$$

implies commutativity of the transfer matrices constructed from  $\mathbb{R}^2$ .

## B Traces

Here we consider calculations of the traces in two basic examples.

### Trace in operator $Q_2$

Let us consider the operator of the following general form

$$\mathbf{A} = P_{10} A_1(z_1, \partial_1|z_0) \cdot P_{20} A_2(z_2, \partial_2|z_0) \cdots P_{N0} A_N(z_N, \partial_N|z_0), \quad (\text{B.1})$$

which acts in the space  $\mathbb{C}[z_0] \otimes \mathbb{C}[z_1] \otimes \cdots \otimes \mathbb{C}[z_N]$ . The operators  $A_k$  acting in the space  $\mathbb{C}[z_0] \otimes \mathbb{C}[z_k]$  are arbitrary functions of the specified arguments. We are going to compute trace of the operator  $\mathbf{A}$  over the space  $\mathbb{C}[z_0]$ . At first step we move permutation operators  $P_{20} \cdots P_{N0}$  to the left and obtain shift operator  $P = P_{12} \cdots P_{1N}$

$$\mathbf{A} = P \cdot P_{10} A_1(z_1, \partial_1|z_2) \cdot A_2(z_2, \partial_2|z_3) \cdots A_N(z_N, \partial_N|z_0)$$

and then we act on  $z_0^n$

$$[\mathbf{A} z_0^n] = P \cdot z_1^n A_1(z_0, \partial_0|z_2) \cdot A_2(z_2, \partial_2|z_3) \cdots A_N(z_N, \partial_N|z_1).$$

The trace of operator is equal to the sum of diagonal matrix elements

$$\text{tr}_0 \mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_0^n [\mathbf{A} z_0^n] \big|_{z_0=0}. \quad (\text{B.2})$$

Applying this general formula we finally obtain

$$\begin{aligned} \text{tr}_0 \mathbf{A} &= \mathbf{P} \cdot \sum_{n=0}^{\infty} \frac{z_1^n}{n!} \partial_0^n A_1(z_0, \partial_0 | z_2) \cdot A_2(z_2, \partial_2 | z_3) \cdots A_N(z_N, \partial_N | z_1) \Big|_{z_0=0} = \\ &= \mathbf{P} \cdot A_1(z_1, \partial_1 | z_2) \cdot A_2(z_2, \partial_2 | z_3) \cdots A_N(z_N, \partial_N | z_0) \Big|_{z_0 \rightarrow z_1} \end{aligned} \quad (\text{B.3})$$

This explicit formula indicates that the trace over infinite dimensional space of operator (B.1) converge without any additional regularization.

### Trace in operator $Q_1$

Consider the formula

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n (B + Az)^n \Phi(z) \Big|_{z=0} &= \sum_{n=0}^{\infty} \partial^n \left[ \sum_{k=0}^n \frac{B^k A^{n-k}}{(n-k)!k!} \sum_{j=0}^{\infty} \frac{1}{j!} \Phi^{(j)}(0) z^{n-k+j} \right] \Big|_{z=0} = \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{n! B^k A^{n-k}}{(n-k)!(k!)^2} \Phi^{(k)}(0) = \sum_{k=0}^{\infty} \frac{B^k \Phi^{(k)}(0)}{k!} \sum_{n=k}^{\infty} \frac{n! A^{n-k}}{(n-k)!k!} = \sum_{k=0}^{\infty} \frac{B^k \Phi^{(k)}(0)}{k! (1-A)^{k+1}}. \end{aligned}$$

Here at the first step the expression in square bracket has been expanded in power series, then the  $n$ -th derivative has been taken and the condition  $z = 0$  has been imposed, which imposes  $j = k$ . After that the order of summations has been changed and the inner sum has been summed up. The last expression is just Taylor series of  $\frac{1}{1-A} \Phi\left(\frac{B}{1-A}\right)$ .

Now we are going to obtain the formula (4.17). The proof is based on the local intertwining relation

$$P_{k0} \lambda^{z_{0k} \partial_0} \cdot P_{k0'} (1 - \frac{1}{\lambda})^{z_{k0'} \partial_k} \cdot P_{00'} \lambda^{z_{00'} \partial_{0'}} = P_{00'} \lambda^{z_{00'} \partial_{0'}} \cdot P_{k0'} \cdot P_{k0} \lambda^{z_{0k} \partial_0} (1 - \frac{1}{\lambda})^{z_{k0} \partial_k}$$

which can be checked straightforwardly. It produces factorization relation for corresponding transfer matrices

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{P} \cdot \mathbf{C} \quad (\text{B.4})$$

where

$$\mathbf{A} \equiv \text{tr}_{\mathbf{V}_0} P_{10} \lambda_1^{z_{01} \partial_0} \cdots P_{N0} \lambda_N^{z_{0N} \partial_0} \quad (\text{B.5})$$

$$\mathbf{B} \equiv \text{tr}_{\mathbf{V}_{0'}} P_{10'} (1 - \frac{1}{\lambda_1})^{z_{10'} \partial_1} \cdots P_{N0'} (1 - \frac{1}{\lambda_N})^{z_{N0'} \partial_N} \quad (\text{B.6})$$

$$\mathbf{C} \equiv \text{tr}_{\mathbf{V}_0} P_{10} \lambda_1^{z_{01} \partial_0} (1 - \frac{1}{\lambda_1})^{z_{10} \partial_1} \cdots P_{N0} \lambda_N^{z_{0N} \partial_0} (1 - \frac{1}{\lambda_N})^{z_{N0} \partial_N} \quad (\text{B.7})$$

Applying (4.15) with  $\lambda \rightarrow 1 - \frac{1}{\lambda}$  we obtain immediately

$$\mathbf{B} \cdot \Psi(\vec{z}) = \mathbf{P} \Psi(\Lambda' \vec{z}).$$

Then we proceed to the calculation of the trace in the definition of  $\mathbf{C}$ . Its building blocks act on the function as follows

$$P_{k0} \lambda^{z_{0k} \partial_0} (1 - \frac{1}{\lambda})^{z_{k0} \partial_k} \Phi(z_k, z_0) = \Phi(z_k, \lambda z_k + \bar{\lambda} z_0).$$

Consequently being applied to the function  $\Psi(\vec{z})$  operator  $\mathbf{C}$  does not change its arguments and only produces overall factor which we calculate using the formula (4.16)

$$\mathbf{C} \cdot \Psi(\vec{z}) = \frac{1}{1 - \bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \Psi(\vec{z}).$$

Finally we act by both sides of (B.4) on the function  $\Psi(\vec{z})$  and obtain

$$\mathbf{A} \cdot \mathbf{P} \Psi(\Lambda' \vec{z}) = \frac{1}{1 - \bar{\lambda}_1 \cdots \bar{\lambda}_N} \cdot \mathbf{P} \Psi(\vec{z})$$

which can be casted in the form (4.17) by means of a linear transformation of  $\vec{z}$ .

## C q-Regularization

In this Appendix we present the necessary modifications of the formulae of sections 3 and 4 after introduction of the  $q$ -regularization. We use the following regularization

$$\mathrm{tr}_0 \mathbf{B} \longrightarrow \mathrm{tr}_0 q^{z_0 \partial_0} \mathbf{B}, \quad |q| < 1$$

Since the boundary conditions change to quasiperiodic ones the shift operator takes the form

$$\mathrm{tr}_0 \left[ q^{z_0 \partial_0} \cdot P_{10} \cdots P_{N0} \right] = P_{12} P_{13} \cdots P_{1N} \cdot \mathrm{tr}_0 \left[ q^{z_0 \partial_0} P_{10} \right] = P \cdot q^{z_1 \partial_1}. \quad (\text{C.1})$$

The introduction of the regularization in the factorization relation (4.8) leads to

$$\begin{aligned} & P q^{z_1 \partial_1} \cdot \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}(u_1, u_2 | v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2 | v_1, v_2) \right] = \\ & = \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}^2(u_1, u_2 | v_2) \cdots \mathbb{R}_{N0}^2(u_1, u_2 | v_2) \right] \cdot \mathrm{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} \mathbb{R}_{10'}^1(u_1 | v_1, u_2) \cdots \mathbb{R}_{N0'}^1(u_1 | v_1, u_2) \right] \end{aligned} \quad (\text{C.2})$$

whereas the second factorization (A.2) modifies in the following manner

$$\begin{aligned} & P q^{z_1 \partial_1} \cdot \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}(u_1, u_2 | v_1, v_2) \cdots \mathbb{R}_{N0}(u_1, u_2 | v_1, v_2) \right] = \\ & = \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}^1(u_1 | v_1, u_2) \cdots \mathbb{R}_{N0}^1(u_1 | v_1, u_2) \right] \cdot \mathrm{tr}_{0'} \left[ q^{z_{0'} \partial_{0'}} \mathbb{R}_{10'}^2(u_1, u_2 | v_2) \cdots \mathbb{R}_{N0'}^2(u_1, u_2 | v_2) \right] \end{aligned} \quad (\text{C.3})$$

The regularized general transfer matrix has the form

$$T_s(u|q) = \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}(u|\ell, s) \mathbb{R}_{20}(u|\ell, s) \cdots \mathbb{R}_{N0}(u|\ell, s) \right]. \quad (\text{C.4})$$

Specifying the parameters in the transfer matrix one obtains the regularized Baxter Q-operators

$$\begin{aligned} Q_1(u - v_1|q) &= \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}^1(u_1 | v_1, u_2) \cdots \mathbb{R}_{N0}^1(u_1 | v_1, u_2) \right], \\ Q_2(u - v_2|q) &= \mathrm{tr}_0 \left[ q^{z_0 \partial_0} \mathbb{R}_{10}^2(u_1, u_2 | v_2) \cdots \mathbb{R}_{N0}^2(u_1, u_2 | v_2) \right]. \end{aligned} \quad (\text{C.5})$$

Regularized commutation relations have the form

$$P q^{z_1 \partial_1} \cdot T_s(u|q) = Q_2(u - s|q) Q_1(u + s + 1|q) = Q_1(u + s + 1|q) Q_2(u - s|q) \quad (\text{C.6})$$

and the set of commutation relations is

$$[P q^{z_1 \partial_1}, Q_1(u|q)] = [P q^{z_1 \partial_1}, Q_2(u|q)] = 0, \quad (\text{C.7})$$

$$[Q_1(u|q), Q_1(v|q)] = [Q_2(u|q), Q_2(v|q)] = [Q_1(u|q), Q_2(v|q)] = 0, \quad (\text{C.8})$$

$$[T_s(u|q), Q_1(v|q)] = [T_s(u|q), Q_2(v|q)] = 0. \quad (\text{C.9})$$

Thus all the formulae of section 4 for different transfer matrices preserve their form provided they have been substituted by their regularized version. It is important that the regularization parameter is the same in all the formulae.

Let us turn to the Baxter equation. In the section 3 we did not use any trace at all so that all modifications are exterior and reduce to simple redefinitions of the main objects. First of all the ordinary transfer matrix  $t(u)$  is changed

$$t(u|q) = \mathrm{tr} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} L_1(u) L_2(u) \cdots L_N(u). \quad (\text{C.10})$$

Further the definition of the  $Q_2$ -operator is changed

$$Q_2(u|q) = P q^{z_1 \partial_1} \cdot R_{12}^2(u) R_{23}^2(u) \cdots R_{N-1,N}^2(u) R_{N0}^2(u) \Big|_{z_0 \rightarrow \frac{z_1}{q}} \quad (C.11)$$

because we want to keep the consistency with representation of this operator as transfer matrix constructed from the  $\mathbb{R}^2$ -operators. The regularized analog of the trace formula in Appendix A has the following form

$$\begin{aligned} \text{tr}_0 \left[ q^{z_0 \partial_0} \cdot P_{10} A_1(z_1, \partial_1 | z_0) \cdot P_{20} A_2(z_2, \partial_2 | z_0) \cdots P_{N0} A_N(z_N, \partial_N | z_0) \right] = \\ = P q^{z_1 \partial_1} \cdot A_1(z_1, \partial_1 | z_2) \cdot A_2(z_2, \partial_2 | z_3) \cdots A_N(z_N, \partial_N | z_0) \Big|_{z_0 = \frac{z_1}{q}} \end{aligned} \quad (C.12)$$

which fixes the needed modifications in definition of the  $Q$ -operator. These changes lead to corresponding modification of the Baxter equation

$$t(u|q) Q_2(u|q) = q \cdot Q_2(u+1|q) + (u_1 u_2)^N \cdot Q_2(u-1|q)$$

due to changes in the main formula (3.4)

$$\begin{aligned} \left( \begin{array}{cc} 1 & 0 \\ -\frac{z_1}{q} & 1 \end{array} \right) \cdot q^{z_1 \partial_1} R_{12}^2(u) R_{23}^2(u) \cdots R_{N0}^2(u) \cdot \left( \begin{array}{cc} q & 0 \\ 0 & 1 \end{array} \right) L_1(u) L_2(u) \cdots L_N(u) \cdot \left( \begin{array}{cc} 1 & 0 \\ z_0 & 1 \end{array} \right) = \\ = q^{z_1 \partial_1} \left( \begin{array}{cc} q & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} R_{12}^2(u+1) & -R_{12}^2(u) \partial_1 \\ 0 & u_1 u_2 R_{12}^2(u-1) \end{array} \right) \cdots \left( \begin{array}{cc} R_{N0}^2(u+1) & -R_{N0}^2(u) \partial_N \\ 0 & u_1 u_2 R_{N0}^2(u-1) \end{array} \right) \end{aligned}$$

and the modification in the formula for the action on the generating function

$$\begin{aligned} Q(u|q) : (1 - x_1 z_1)^{2\ell} \cdots (1 - x_N z_N)^{2\ell} \mapsto \\ \mapsto (1 - q x_1 z_N)^{\ell-u} (1 - x_1 z_1)^{\ell+u} \cdots (1 - x_N z_{N-1})^{\ell-u} (1 - x_N z_N)^{\ell+u}. \end{aligned} \quad (C.13)$$

Let us consider the modifications in the uniform approach to the derivation of Baxter equation for both  $Q$ -operators. It turns out that the Baxter equations are different for  $q$ -regularized operators  $Q_1$  and  $Q_2$ . The transfer matrix which is constructed as the trace over finite dimensional space  $\mathbb{V}_n$  has the form

$$t_n(u|q) = \text{tr} q^{z_0 \partial_0} \mathbf{R}_{10}(u|\ell, \frac{n}{2}) \mathbf{R}_{20}(u|\ell, \frac{n}{2}) \cdots \mathbf{R}_{N0}(u|\ell, \frac{n}{2}). \quad (C.14)$$

Because

$$\mathcal{D} q^{z_0 \partial_0} = q^{n+1} \cdot q^{z_0 \partial_0} \mathcal{D},$$

we have the modification <sup>4</sup>

$$\begin{aligned} T_{\frac{n}{2}}(u) = t_n(u) + q^{n+1} \cdot T_{-\frac{n}{2}-1}(u). \\ \text{tr} \mathcal{D} q^{z_0 \partial_0} \mathbb{R}_{10}(u|\ell, \frac{n}{2}) \mathbb{R}_{20}(u|\ell, \frac{n}{2}) \cdots \mathbb{R}_{N0}(u|\ell, \frac{n}{2}) \mathcal{D}^{-1} = q^{n+1} \cdot T_{-\frac{n}{2}-1}(u|q) \end{aligned}$$

and the determinant relation takes the following modified form

$$P q^{z_1 \partial_1} \cdot t_n(u|q) = \begin{vmatrix} Q_1(u + \frac{n}{2} + 1|q) & q^\beta \cdot Q_2(u + \frac{n}{2} + 1|q) \\ q^\alpha \cdot Q_1(u - \frac{n}{2}|q) & Q_2(u - \frac{n}{2}|q) \end{vmatrix}, \quad \text{where } n+1 = \alpha + \beta.$$

The general bilinear relation involving  $Q_1$  is derived starting from the following determinant

$$\begin{vmatrix} Q_1(a|q) & q^{n+1} Q_2(a|q) & Q_1(a|q) \\ Q_1(b|q) & Q_2(b|q) & Q_1(b|q) \\ q^{m+1} Q_1(c|q) & Q_2(c|q) & q^{m+1} Q_1(c|q) \end{vmatrix} = 0$$

---

<sup>4</sup>Now we change normalizations of operators  $R$  and  $R^1$  as in the subsection 4.2 in order to deal with Baxter relations.

Specifying parameters

$$a = u + \frac{n}{2} + 1 ; b = u - \frac{n}{2} ; c = u - m - \frac{n}{2} - 1 ,$$

we arrive at the relation

$$\begin{aligned} & t_m \left( u - 1 - \frac{n+m}{2} | q \right) \cdot Q_1 \left( u + 1 + \frac{n}{2} | q \right) - t_{n+m+1} \left( u - \frac{m+1}{2} | q \right) \cdot Q_1 \left( u - \frac{n}{2} | q \right) + \\ & + q^{m+1} \cdot t_n (u | q) \cdot Q_1 \left( u - 1 - m - \frac{n}{2} | q \right) = 0 . \end{aligned}$$

In a special case  $n = m = 0$  one obtains the Baxter equation for the operator  $Q_1(u | q)$

$$t(u | q) Q_1(u | q) = Q_1(u + 1 | q) + q \cdot (u_1 u_2)^N \cdot Q_1(u - 1 | q) .$$

For completeness we derive the Baxter equation for the second operator  $Q_2$ . All the difference is in the starting point

$$\begin{vmatrix} Q_1(a | q) & q^{n+1} Q_2(a | q) & q^{n+1} Q_2(a | q) \\ Q_1(b | q) & Q_2(b | q) & Q_2(b | q) \\ q^{m+1} Q_1(c | q) & Q_2(c | q) & Q_2(c | q) \end{vmatrix} = 0 .$$

Next we specify parameters in the same way and arrive at the relation

$$\begin{aligned} & q^{n+1} \cdot t_m \left( u - 1 - \frac{n+m}{2} | q \right) \cdot Q_2 \left( u + 1 + \frac{n}{2} | q \right) - t_{n+m+1} \left( u - \frac{m+1}{2} | q \right) \cdot Q_2 \left( u - \frac{n}{2} | q \right) + \\ & + t_n (u | q) \cdot Q_2 \left( u - 1 - m - \frac{n}{2} | q \right) = 0 , \end{aligned}$$

and at the special point  $n = m = 0$  one obtains the Baxter equation (5.29) for the second operator.

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